Lecture 4
Practical Aircraft Aeroelasticity

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3 assumptions by Theodorsen:

- Attached flow \(\rightarrow\) small motion
- Airfoil = flat plate
- Flat wake

\[ \rightarrow \text{Unsteady circulatory lift :} \]

\[ l_c = \pi \rho U c C(k) \left( U \alpha + \dot{h} + \left( \frac{3c}{4} - x_f \right) \dot{\alpha} \right) \]

Assuming sinusoidal motion

\[ \alpha = \alpha_0 e^{j\omega t} \]

\[ h = h_0 e^{j\omega t} \]

\[ \rightarrow C(k) = 1 - \frac{0.165}{1 - \frac{0.0455}{k} j} - \frac{0.335}{1 - \frac{0.3}{k} j} \]
Non-sinusoidal motion

• Theodorsen analysis assumes sinusoidal motion
  ➔ Equations of motion are only valid at zero airspeed or at the flutter condition.
• But they are also valid in the case of forced sinusoidal excitation.

➔ Objective:
Calculate the response of an aeroelastic system with Theodorsen aerodynamics to any excitation force
Forced response

• Until now, unforced motion only:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \dddot{\alpha}
\end{pmatrix} +
\begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix} =
\begin{pmatrix}
  -l(t) \\
  m(t)
\end{pmatrix}
\]

• Consider an external loads (e.g. control input) applied on one of the degrees of freedom

  – Force on plunge:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \dddot{\alpha}
\end{pmatrix} +
\begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix} +
\begin{pmatrix}
  l(t) \\
  -m(t)
\end{pmatrix} =
\begin{pmatrix}
  F_h(t) \\
  0
\end{pmatrix}
\]

  – Moment on pitch:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \dddot{\alpha}
\end{pmatrix} +
\begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix} +
\begin{pmatrix}
  l(t) \\
  -m(t)
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  M_\alpha(t)
\end{pmatrix}
\]
Sinusoidal force

- Using Theodorsen analysis, the force must be sinusoidal:
  \[ F_h = F_0 e^{j\omega t} \]
  \[ M_\alpha = M_0 e^{j\omega t} \]

- Once all the transients have died out, the steady-state response of the pitch-plunge wing will also be sinusoidal:
  \[ h = h_0 e^{j\omega t} \]
  \[ \alpha = \alpha_0 e^{j\omega t} \]

→ Replace \( F_h \) (or \( M_\alpha \)) and \( h \) (or \( \alpha \)) in the EoM
Forced equations of motion

Force in plunge:

\[
\begin{pmatrix}
K_h - \omega^2 m + \pi \rho U c C(k) j \omega \\
-\omega^2 \rho \pi b \\
-\omega^2 S - \pi \rho U e c C(k) j \omega \\
+ \left( x_f - \frac{c}{2} \right) \rho \pi b^2 \omega^2
\end{pmatrix}
\begin{pmatrix}
-\omega^2 S + \rho \pi b^2 U j \omega + \rho \pi b^2 \left( x_f - \frac{c}{2} \right) \omega^2 \\
+ \pi \rho U c C(k) \left( U + \left( \frac{3}{4} c - x_f \right) j \omega \right) \\
K_a - \omega^2 I_a + \left( \frac{3}{4} c - x_f \right) \rho \pi b^2 U j \omega \\
-\pi \rho U e c C(k) \left( U + \left( \frac{3}{4} c - x_f \right) j \omega \right)
\end{pmatrix}
\begin{pmatrix}
\frac{h_0}{\omega b/U} \\
\frac{\alpha_0}{\omega b/U}
\end{pmatrix} = \begin{pmatrix} F_0 \end{pmatrix}
\]

with \( k = \omega b/U \)
This equation is of the form

$$H(\omega)q_0 = F$$

where $H^{-1}(\omega)$ is the Frequency Response Function Matrix

→ Knowing $H^{-1}(\omega)$ and $F \rightarrow q_0 = H(\omega)^{-1}F$

- $H$ is a function of $\omega$
- Response amplitude $q_0$ is also a function of $\omega$
- The response is still sinusoidal.
- What happens if $F$ is also a function of $\omega$?
Fourier Transforms

• Using Fourier analysis, any force time signal $F_h(t)$ can be transformed to the frequency domain and written as:

$$F_0(\omega)e^{j\omega t}$$

• The signal that is of particular interest here is the impulse. Its Fourier Transform is

$$F_0(\omega) = 1$$

→ If we set $F_0(\omega) = 1$ and then calculate

$$q_0(\omega) = H^{-1}(\omega)F = H^{-1}(\omega)$$

The resulting response amplitude $q_0(\omega)$, will be the Frequency Response Function (FRF).
FRF for pitch-plunge system

FRF of $h$

The two modes are clearly present

FRF of $\alpha$

The first mode is present as an anti-resonance
• **Impulse Response Function (IRF)** = inverse Fourier Transform of the FRF

• From the knowledge of the FRF, $q_0(\omega)$, we can simply apply the inverse Fourier Transform to it and calculate the IRF.

• The IRF is a time domain signal and it is not sinusoidal:
  - For a **dynamic system**, the IRF is typically an exponentially decaying sinusoid.
  - For a **fluttering aeroelastic system**, the IRF is an exponentially increasing sinusoid.
Impulse response of pitch-plunge airfoil

\[ U = 15 \text{ m/s, } x/c = 0.4 \]

\[ U = 25 \text{ m/s, } x/c = 0.4 \]

\[ V < V_F \]

\[ V > V_F \]
Damped sinusoidal motion

• Until now, we showed:
  – Theodorsen theory is only valid for sinusoidal motion.
  – But it can also be used to calculate impulse responses.
  – The form of the impulse response can tell is if the system is stable or unstable.

• Stability analysis is slow and can be less accurate when performed on impulse responses.

→ We need a method for calculating the damping at all airspeeds directly from the EoM
The p-k Method

= most popular technique for obtaining aeroelastic solutions from Theodorsen-type aeroelastic systems

• Proposed in the 80’s and since then has become the industrial standard

• Virtually all aircraft flying today have been designed using the p-k method
Basics

- The p-k method uses the structural equations of motion in the standard form

\[
\begin{pmatrix}
m & S \\
S & I_\alpha \\
\end{pmatrix}
\begin{pmatrix}
\ddot{h} \\
\ddot{\alpha} \\
\end{pmatrix}
+ \begin{pmatrix}
K_h & 0 \\
0 & K_\alpha \\
\end{pmatrix}
\begin{pmatrix}
h \\
\alpha \\
\end{pmatrix}
= \begin{pmatrix}
-l(t) \\
m(t) \\
\end{pmatrix}
\]

- Coupled with Theodorsen aerodynamic forces of the form

\[
l(t) = \left\{ \rho \pi b^2 U j \omega \alpha_0 + \pi \rho U c C(k) \left( U \alpha_0 + j \omega h_0 + \left( \frac{3}{4} c - x_f \right) j \omega \alpha_0 \right) \right. \\
+ \rho \pi b^2 \left( -\omega^2 h_0 + \left( x_f - \frac{c}{2} \right) \omega^2 \alpha_0 \right) \right\} e^{j \omega t}
\]

with \( k = \omega b / U \)
Time-Frequency domain

• Coefficients of EoM are frequency dependent
• EoM = Time-frequency domain equations.

\[
\begin{pmatrix}
m & S \\
S & I_a
\end{pmatrix}
\begin{pmatrix}
h \\
\dot{\alpha}
\end{pmatrix} + \begin{pmatrix}
K_h & 0 \\
0 & K_a
\end{pmatrix} \begin{pmatrix}
h \\
\alpha
\end{pmatrix} - \frac{1}{2} \rho U^2 \begin{pmatrix}
-4\pi C(k) jk + 2\pi k^2 \\
-2\pi c C(k) - 2\pi b jk - 4\pi C(k) \left(\frac{3}{4} c - x_f\right) jk - 2\pi b^2 k^2 \\
4\pi ec C(k) jk - 2\pi \left(x_f - \frac{c}{2}\right) k^2 \\
2\pi ec^2 C(k) - 2\left(\frac{3}{4} c - x_f\right) \pi b jk \\
+4\pi ec C(k) \left(\frac{3}{4} c - x_f\right) jk + 2\pi \left(x_f - \frac{c}{2}\right)^2 k^2 + \pi \frac{b^2}{4} k^2
\end{pmatrix} \begin{pmatrix}
h \\
\alpha
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

• General form:

\[
M_s \ddot{q} + K_s q - \frac{1}{2} \rho U^2 Q(k) q = 0
\]
• Solving time-frequency equations of motion is a nonlinear eigenvalue problem.

• The basis of the p-k method is to define

\[ p = \frac{d}{dt} \]

so that \( \ddot{q} = p^2 q \)

• Re-writing the equations of motion as

\[
M_s \ddot{q} + K_s q - \frac{1}{2} \rho U^2 Q(k) q = 0 \quad \rightarrow \quad \left( p^2 M_s + K_s - \frac{1}{2} \rho U^2 Q(k) \right) q = 0
\]
These equations of motion are clearly an eigenvalue problem. For a non-trivial solution:

$$p^2 I + M_s^{-1} \left( K_s - \frac{1}{2} \rho U^2 Q(k) \right) = 0 \quad (1)$$

= 1 equation with 2 unknowns ($p$ and $k$) → Need a second condition.

Note that $p$ is an eigenvalue of the matrix

$$M_s^{-1} \left( K_s - \frac{1}{2} \rho U^2 Q(k) \right)$$
Eigenvalue estimation (2)

- $k$ is a reduced frequency: $k = \frac{\omega b}{U}$
- Recall that the magnitude of the eigenvalue is the frequency: $|p| = \omega$

$\rightarrow$ The second equation is: $|p| = k \frac{U}{b}$ (2)

$\rightarrow$ The p-k method consists in:
Finding an eigenvalue and a frequency that satisfy both equations (1) and (2).
The p-k solution

The solution of these equations is iterative:

1. Guess a value for the frequency $k$
2. Calculate $p$ from the resulting eigenvalue problem (1):
   \[
   p^2 I + M_s^{-1} \left( K_s - \frac{1}{2} \rho U^2 Q(k) \right) = 0
   \]
3. $p$ and $k$ should satisfy equation (2):
   \[
   |p| = k \frac{U}{b}
   \]
4. If they do not, we change the value of $k$ and re-calculate $p$ until equation (2) is satisfied.

= Frequency matching
Frequency matching

1. Choose an airspeed, $U$.

2. Choose the $i$th degree of freedom ($i = 1$ denotes plunge, $h$, $i = 2$ denotes pitch, $\alpha$).

3. Set an initial value for $\omega$ and, hence, $k$. This value can be $\sqrt{K_h/m}$ if $i = 1$ or $\sqrt{K_\alpha/I_\alpha}$ if $i = 2$.

4. Calculate the eigenvalues $p$ of equation

$$p^2 I + M^{-1} \left( K - \frac{1}{2} \rho U^2 Q(k) \right) = 0$$

5. Sort the eigenvalues in ascending order of imaginary part.

6. Choose a new value of $\omega$ equal to the imaginary part of the $i$th sorted eigenvalue and repeat from step 4.

7. When the value of $\omega$ converges stop the iteration. Store the converged value of the $i$th eigenvalue.

8. Repeat from step 2 for all the $i$ degrees of freedom.

9. Repeat from step 1 for other airspeeds.
p-k method characteristics

- **Converges very quickly** to the correct eigenvalue.
- **Suitable for** large computational problems.
- Calculates complete eigenvalues and therefore damping ratios and natural frequencies.
- **Valid at all airspeeds**, not just the flutter speed.
- Estimated flutter speeds are very similar to those obtained from the flutter determinant solution.
Sample result

NACA0012 airfoil, \( x_f = 0.4c \)

- Wagner Function
- p-k Method

Damping Ratio vs. Airspeed (m/s)
Sample result
Roger’s Approximation

= Alternative to solve the p-k equations
→ Transformation of EoM into the time domain using Roger’s Approximation.

\[ M_s \ddot{q} + K_s q - \frac{1}{2} \rho U^2 Q(k) q = 0 \]

The frequency-dependent part of equations of motion, \( Q(k) \), is approximated as:

\[ Q(k) = A_0 + A_1 jk + A_2 (jk)^2 + \sum_{n=1}^{n_l} A_{2+n} \frac{jk}{jk + \gamma_n} \]  

(3)

where \( n_l \) is the number of aerodynamic lags
\( \gamma_n \) are aerodynamic lag coefficients.
Roger’s EOMs

• The equations of motion of the complete aeroelastic system become:

\[
\dot{\mathbf{q}} = \begin{pmatrix}
-\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{A}_3 & \ldots & -\mathbf{M}^{-1}\mathbf{A}_{n+2} \\
\mathbf{I} & 0 & 0 & \ldots & 0 \\
0 & \mathbf{I} & -V\gamma_1/b\mathbf{I} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \mathbf{I} & 0 & \ldots & -V\gamma_n/b\mathbf{I}
\end{pmatrix} \begin{pmatrix}
\mathbf{q}
\end{pmatrix}
\]

where

\[
\mathbf{M} = \mathbf{M}_s - \frac{1}{2}\rho b^2 \mathbf{A}_2, \quad \mathbf{C} = \mathbf{C}_s - \frac{1}{2}\rho Ub\mathbf{A}_1, \quad \mathbf{K} = \mathbf{K}_s - \frac{1}{2}\rho U^2\mathbf{A}_0, \quad \mathbf{A}_j = -\frac{1}{2}\rho U^2\mathbf{A}_j
\]

• Usually:

\[
n_l = 4, \quad \gamma_n = -1.7k_{\text{max}} \frac{n}{(n_l + 1)^2}, \quad k_{\text{max}} = \text{maximum } k \text{ of interest}
\]
Roger details

• Matrices $A_0, A_1, \ldots$ are obtained from equation (3) by performing a least-squares curve fit of $Q(k)$ for several values of $k$.

• If the aeroelastic system has $n$ degrees of freedom, Roger’s equations of motion will have $(n_l+2)n$ states. Of those, $n_l n$ states are aerodynamic states.

• There are similar but more efficient schemes for improving the curve fit of $Q(k)$ and reducing $n_l$, e.g. the minimum state method.
Practical Aeroelasticity

• For an aircraft, the matrix $Q(k)$ is obtained using a panel method-based aerodynamic model.
• Usually performed by means of commercial packages, such as MSC.Nastran or Z-Aero.
• For a chosen set of $k$ values, e.g. $k_1, k_2, \ldots, k_m$, the corresponding $Q$ matrices are returned.
• Values of $Q$ at intermediate $k$ values are obtained by interpolation.
• p-k method $\rightarrow$ flutter solution, time-domain responses
3D wings

• All the methods described until now concern airfoil sections (2D)
• These results must now be extended to 3D wings because all wings are 3D
• Two methods for 3D wing aeroelasticity:
  – Strip theory
  – Panel methods
Strip theory

→ Dividing the wing into spanwise small strips

The instantaneous lift and moment acting on each strip

2D sectional lift and moment theories (quasi-steady, unsteady, ..)
Panel methods

→ Wing is replaced by its camber surface.

Camber surface itself is replaced by panels of mathematical singularities, solutions of Laplace’s equation
Hancock Model

= Simple 3D wing model

- Rigid flat plate
- Dimensions
  - span $s$
  - chord $c$
  - thickness $t$
- Suspended through an axis $x_f$ by two torsional springs:
  - in roll ($K_\gamma$)
  - in pitch ($K_\theta$)
Equations of motion

- As with the 2D pitch plunge wing, the equations of motion are derived using energy considerations.
- The kinetic energy of a small mass element $dm$ of the wing is given by

$$dT = \frac{1}{2} \dot{z}^2 dm = \frac{1}{2} dm \left( y \ddot{y} + (x - x_f) \dot{\theta} \right)^2$$

- The total kinetic energy of the wing is:

$$T = \frac{m}{12} \left( 2s^2 \dot{\gamma}^2 + 3s (c - 2x_f) \ddot{\gamma} \dot{\theta} + 2 \left( c^2 - 3x_fc + 3x_f^2 \right) \dot{\theta}^2 \right)$$
Structural equations

- The potential energy of the wing is simply

\[ V = \frac{1}{2} K_\gamma \gamma^2 + \frac{1}{2} K_\theta \theta^2 \]

- The full structural equations of motion are then:

\[
\begin{pmatrix}
I_\gamma & I_{\gamma\theta} \\
I_{\gamma\theta} & I_\theta
\end{pmatrix}
\begin{bmatrix}
\ddot{\gamma} \\
\ddot{\theta}
\end{bmatrix}
+
\begin{pmatrix}
K_\gamma & 0 \\
0 & K_\theta
\end{pmatrix}
\begin{bmatrix}
\gamma \\
\theta
\end{bmatrix}
=
\begin{bmatrix}
M_1 \\
M_2
\end{bmatrix}
\]

\[ I_\gamma = \frac{ms^2}{3}, \quad I_{\gamma\theta} = m\left(c - 2x_f\right)s/4, \quad I_\theta = m\left(c^2 - 3x_f c + 3x_f^2\right)/3 \]
Aerodynamic model

Using the strip-theory

– 2D aerodynamics: quasi-steady or unsteady approximations for the lift and moment around the flexural axis are applied to infinitesimal strips of wing

– Integration of lift and moment over the entire span of the wing

→ Quasi-steady pseudo-3D lift and moment acting on the Hancock wing:

\[ M_1 = -\int_0^s yl(y)dy \]
\[ M_2 = -\int_0^s m_{xf}(y)dy \]
Quasi-steady strip theory

Notations: $\theta = \alpha$ (pitch) and $h = y\gamma$ (flap)

\[
I = \left\{ \rho \pi b^2 \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \dot{\theta} \right) + \rho \pi b^2 U \dot{\theta} + \rho U^2 c \pi \left( \theta(t) + \frac{y \dot{\gamma}(t)}{U} + \left( \frac{3}{4} c - x_f \right) \frac{\dot{\theta}(t)}{U} \right) \right\} dy
\]

\[
m_{xf} = \left\{ \rho \pi b^2 \left( x_f - \frac{c}{2} \right) \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \dot{\theta} \right) - \frac{\rho \pi b^4}{8} \ddot{\theta} - \left( \frac{3}{4} c - x_f \right) \rho \pi b^2 U \dot{\theta} + \rho U^2 c^2 \pi \left( \theta(t) + \frac{y \dot{\gamma}(t)}{U} + \left( \frac{3}{4} c - x_f \right) \frac{\dot{\theta}(t)}{U} \right) - \frac{1}{16} \rho U c^3 \pi \dot{\theta} \right\} dy
\]

→ Strip theory integrations will yield the total moments around the $y = 0$ and $x = x_f$ axes.
3D Quasi-steady equations of motion

Full 3D quasi-steady equation of motion:

\[
\begin{align*}
\begin{pmatrix}
I_\gamma & I_\gamma \\
I_\gamma & I_\theta \\
\end{pmatrix} & + \rho \pi b^2 s \begin{pmatrix}
s^2/3 & -(x_f - c/2)s/2 \\
-(x_f - c/2)s/2 & (x_f - c/2)^2 + \frac{b^2}{8} \\
\end{pmatrix} \begin{Bmatrix}
\dot{\gamma} \\
\dot{\theta} \\
\end{Bmatrix} + \\
\rho U c \pi s & \begin{pmatrix}
s^2/3 & \frac{1}{2}s \left(\frac{3}{4}c - x_f\right) + \frac{c}{4} \\
-ecs/2 & \left(\frac{3}{4}c - x_f\right) \left(\frac{c}{2} - x_f\right) + \frac{c^2}{16} \\
\end{pmatrix} \begin{Bmatrix}
\dot{\gamma} \\
\dot{\theta} \\
\end{Bmatrix} + \\
\begin{pmatrix}
K_\gamma & 1/2 \rho U^2 c \pi s^2 \\
0 & K_\theta - \rho U^2 ec^2 \pi s \\
\end{pmatrix} \begin{Bmatrix}
\gamma \\
\theta \\
\end{Bmatrix} = \begin{Bmatrix}
0 \\
0 \\
\end{Bmatrix}
\end{align*}
\]
Natural frequencies and damping ratios
Theodorsen function aerodynamics (unsteady frequency domain) can be implemented directly using strip theory:

\[
\begin{align*}
    l &= \left\{ \rho \pi U \alpha C(k) \left( U \theta + y \dot{\gamma} + \left( \frac{3}{4} c - x_f \right) \dot{\theta} \right) + \rho \pi b^2 \dot{\theta} + \\
       &\quad \rho \pi b^2 \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \ddot{\theta} \right) \right\} \\
    m_{xf} &= \left\{ \rho \pi U \alpha C(k) \left( U \theta + y \dot{\gamma} + \left( \frac{3}{4} c - x_f \right) \dot{\theta} \right) - \left( \frac{3}{4} c - x_f \right) \rho \pi b^2 \dot{\theta} + \\
           &\quad \rho \pi b^2 \left( x_f - \frac{c}{2} \right) \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \ddot{\theta} \right) - \rho \pi \frac{b^4}{8} \ddot{\theta} \right\}
\end{align*}
\]
Flutter determinant

Flutter determinant for the Hancock model

\[
Q = \begin{pmatrix}
K_\gamma + \frac{1}{3} \omega \pi \rho s^3 \left( j U c C(k) - \omega b^2 \right) & \frac{1}{2} \pi \rho s^2 \left\{ U^2 c C(k) \left( 1 + j \omega \left( \frac{3}{4} c - x_f \right) \right) + \omega b^2 \left( U j + \omega \left( x_f - \frac{c}{2} \right) \right) \right\} \\
\frac{1}{2} \pi \rho s^2 \left( - j U e c^2 C(k) + b^2 \left( x_f - \frac{c}{2} \right) \right) & \pi \rho s \left\{ - U^2 e c^2 C(k) + j \omega U \left( \frac{3}{4} c - x_f \right) \left( e c^2 C(k) + b^2 \right) - \omega^2 \left( \frac{b^4}{8} + b^2 \left( x_f - \frac{c}{2} \right)^2 \right) \right\} + K_\theta
\end{pmatrix}
\]

To solve similarly than the 2D pitch-plunge airfoil (flutter determinant)
p-k solution

Instead of solving the flutter determinant, we can apply the p-k method. The process is exactly as presented in the 2D case.
Comparison of flutter speeds

Wagner and Theodorsen solutions are identical.

Quasi-steady solution is the most conservative

(ignore the other solutions)
Comparison of 3D and strip theory, static case

The 3D lift distribution is completely different to the strip theory result!

Strip theory:

\[ C_L = 2\pi\alpha \]

Lifting line method:

\[ C_L = \frac{2\pi AR}{AR + 2} \alpha \]

- Strip theory is only exact when the wing’s aspect ratio is infinite.
- It becomes completely unsatisfactory at moderate and small aspect ratios (less than 8-10).
Vortex Lattice Method

Dividing the wing planform into panels on which lie vortex rings, usually called a vortex lattice

**Vortex ring =**

- Rectangle made up of four straight line vortex segments.
- Elementary solution of Laplace’s equation.
- $n_{ij}$, normal to the ring, at its midpoint
- Circulation, $\Gamma$, is constant over all the 4 segments of a vortex ring.
- Each segment induces a velocity $[u \ v \ w]$ at a general point $P$.
- Self-induced velocity = 0
  
  If point $P$ lies on a vortex ring segment, the velocity is 0.
Panelling up and solving

- The wing can be swept, tapered and twisted.

- It **cannot have thickness**; the panels lie on the wing’s camber surface.

- The wake must also be panelled up.

- Objective of VLM: calculation of the values of the circulations $\Gamma$ on each wing panel at each instance in time.

- The circulation of the **wake panels** does not change in time. Only the vorticities of the **wing panels** are unknowns.
Boundary conditions

Two boundary conditions:

- **Impermeability**
  
  ‘No flow normal to the panel at its collocation point’
  
  → This conditions gives all the $\Gamma$ values.

- **Kutta condition**
  
  ‘Flow separates at the trailing edge’
  
  → Automatically satisfied by placing the leading edge of each vortex ring on the quarter-chord of its corresponding geometric panel.
Calculating forces

• Once the vorticities on the wing panels are known, the lift and moment acting on the wing can be calculated

• These are calculated from the pressure difference acting on each panel

• Summing the pressure differences of the entire wing yields the total forces and moments
Panels for static wing

Even if the wing is not moving → 3D wing tip effects
(wing tip vortices, induced drag)
→ Wake must be modelled
Wake shapes behind a rectangular wing (AR=4, AOA=5°) that underwent an impulsive start from rest.
Effect of Aspect Ratio on lift coefficient

3D results approach Wagner’s function (2D result) as the AR increases.
Free wakes

- Free wake behind a flapping goose wing.
- Airspeed: 18m/s.
Industrial use

• **Unsteady wakes** are beautiful but expensive to calculate.

• For practical purposes, a **fixed wake** is used with unsteady vorticity

  → Theodorsen’s method.

• The wake propagates at the free stream airspeed and in the free stream direction.

• Only a short length of the wake is simulated (a few chord-lengths).

• The result is a **linearized aerodynamic model**
Commercial packages

• There are two major commercial packages that can calculate 3D unsteady aerodynamics using panel methods:
  – MSC.Nastran (MSC Software)
  – ZAERO (ZONA Technology)

• They can both deal with complex aircraft geometries.
The examples manual of ZAERO features an Advanced Fighter Aircraft (AFA) model.
BAH Example

- Bisplinghoff, Ashley and Halfman wing

- FEM with 12 nodes and 72 dof
BAH Example

First 5 modes of BAH wing (US $\rightarrow$ inversed signs !)

'\textit{hump}' mode
GTA Example

= Very simple aeroelastic model of a Generic Transport Aircraft (GTA)

Finite element model: Bar elements with 678 degrees of freedom

Aerodynamic model: 2500 doublet lattice panels
First 7 flexible modes:

Clear flutter mechanism between first (first wing bending) and third mode (aileron deflection)
Time domain plots for the GTA

$V < V_F$

$V = V_F$
SuperSonic Transport

- SST = Concorde and Tupolev Tu-144
- The aeroelastic model is a half-model
- The aerodynamics features the wing and a rectangle for the wall
Flutter plots for SST

First 9 flexible modes:

Clear flutter mechanism between first and third mode.
Summary

• Aeroelastic design in industry is almost exclusively carried out using lattice methods (vortex or doublet) combined with a Finite Element model with few retained modes (generally fewer than 100).

• Panel methods (source and doublet) are also used for more detailed geometric representation, including thickness.

• The aerodynamic forces are written in the form of Aerodynamic Force Coefficient Matrices that depend on frequency.

• The flutter solution is obtained using the **p-k method** = Frequency matching