Aeroelasticity

Lecture 5:
Practical Aircraft Aeroelasticity

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Introduction

- In the previous lecture we showed that Theodorsen theory can be used to obtain an algebraic system of two equations for the two unknowns:
  - Flutter frequency
  - Flutter airspeed

\[
\begin{pmatrix}
K_h - \omega^2 m + \pi \rho U c C(k) j \omega & -\omega^2 S + \rho \pi b^2 U j \omega + \rho \pi b^2 \left(x_f - \frac{c}{2}\right) \omega^2 \\
-\omega^2 \rho \pi b^2 & +\pi \rho U c C(k) \left(U + \left(\frac{3}{4}c - x_f\right) j \omega\right)
\end{pmatrix}
\begin{pmatrix}
h_0 \\
a_0
\end{pmatrix} = 0
\]
Non-sinusoidal motion

- This algebraic system of equations is only valid if the system is undergoing purely sinusoidal oscillations. There are three situations in which this can happen:
  - At the flutter condition
  - At wind-off conditions if the structural damping is zero.
  - At any airspeed if the wind response to an external sinusoidal load.
- The last situation means that we can calculate the response of an aeroelastic system with Theodorsen aerodynamics to any excitation force.
- This is particularly useful because it will allow us to calculate the natural frequencies and damping ratios at all airspeeds, as we did with time-domain analysis.
Forced response

• Until now we have only considered unforced motion:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \ddot{\alpha}
\end{pmatrix}
+ \begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix}
= \begin{pmatrix}
  -l(t) \\
  m(t)
\end{pmatrix}
\]

• No we will consider an external loads (e.g. control input) applied on one of the degrees of freedom. E.g. force on plunge:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \ddot{\alpha}
\end{pmatrix}
+ \begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix}
+ \begin{pmatrix}
  l(t) \\
  -m(t)
\end{pmatrix}
= \begin{pmatrix}
  F_h(t) \\
  0
\end{pmatrix}
\]

• Or moment on pitch:

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \ddot{\alpha}
\end{pmatrix}
+ \begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix}
+ \begin{pmatrix}
  l(t) \\
  -m(t)
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  M_\alpha(t)
\end{pmatrix}
\]
Sinusoidal force

• If Theodorsen analysis is to be applied, the force must be sinusoidal, e.g.

\[ F_h = F_0 e^{j\omega t} \]

\[ M_\alpha = M_0 e^{j\omega t} \]

• The steady-state response of the pitch-plunge wing will also be sinusoidal once all the transients have died out, i.e.

\[ \alpha = \alpha_0 e^{j\omega t} \]

\[ h = h_0 e^{j\omega t} \]

• Substituting into the complete equations of motion:
Forced equations of motion

- **Force in plunge**

\[
\begin{pmatrix}
K_h - \omega^2 m + \pi \rho U c C(k) j\omega \\
-\omega^2 \rho \pi b \\
-\omega^2 S - \pi \rho U c C(k) j\omega \\
+\left(x_f - \frac{c}{2}\right) \rho \pi b^2 \omega^2
\end{pmatrix}
= \begin{pmatrix}
-\omega^2 S + \rho \pi b^2 U j\omega + \rho \pi b^2 \left(x_f - \frac{c}{2}\right) \omega^2 \\
+\pi \rho U c C(k) \left(U + \left(\frac{3}{4} c - x_f\right) j\omega\right) \\
K_a - \omega^2 I_a + \left(\frac{3}{4} c - x_f\right) \rho \pi b^2 U j\omega \\
-\pi \rho U c C(k) \left(U + \left(\frac{3}{4} c - x_f\right) j\omega\right)
\end{pmatrix}
\begin{pmatrix}
h_0 \\
\alpha_0
\end{pmatrix}
= \begin{pmatrix}
F_0 \\
0
\end{pmatrix}
\]

With \( k = \omega b / U \)
FRF Matrix

• This equation is of the form

\[ H(\omega)q_0 = F \]

• where \( H^{-1}(\omega) \) is the Frequency Response Function Matrix.

• Note that, since \( H \) is is a function of \( \omega \), the response amplitude \( q_0 \) will also be a function of \( \omega \).

• The response is still sinusoidal. But what happens if \( F \) is also a function of \( \omega \)?
Fourier Transforms

• Using Fourier analysis, any force time signal $F_h(t)$ can be transformed to the frequency domain and written as:

$$F_0(\omega)e^{j\omega t}$$

• The signal that is of particular interest here is the impulse. Its Fourier Transform is

$$F_0(\omega)=1$$

• This means that if we set $F_0(\omega)=1$ and then calculate

$$q_0(\omega)=H^{-1}(\omega)F$$

• the resulting response amplitude $q_0(\omega)$, will be the Frequency Response Function (FRF).
FRF for pitch-plunge system

FRF of $h$

The two modes are clearly present

FRF of $\alpha$

The first mode is present as an anti-resonance
Impulse Response Function

• The Impulse Response Function (IRF) is the inverse Fourier Transform of the FRF.
• As we have calculated the FRF, $q_0(\omega)$, we can simply apply the inverse Fourier Transform to it and calculate the IRF.
• The IRF is a time domain signal and it is not sinusoidal. For a dynamic system, the IRF is typically an exponentially decaying sinusoid.
• For a fluttering aeroelastic system, the IRF is an exponentially increasing sinusoid.
Impulse response of pitch-plunge airfoil

U=15m/s, x/c=0.4

U=25m/s, x/c=0.4

V=15m/s

V=25m/s
Damped sinusoidal motion

• The previous discussion shows that:
  – Theodorsen aerodynamics is only valid for sinusoidal motion.
  – Yet Theodorsen aerodynamics can also be used to calculate impulse responses.
  – The form of the impulse response can tell is if the system is stable or unstable.

• Stability analysis is slow and can be less accurate when performed on impulse responses.

• We need a method for calculating the damping at all airspeeds directly from the equations of motion.
The p-k Method

- The p-k method is the most popular technique for obtaining aeroelastic solutions from Theodorsen-type aeroelastic systems.
- It was proposed in the 80s and since then has become the industrial standard.
- Virtually all aircraft flying today have been designed using the p-k method.
Basics

• The p-k method uses the structural equations of motion in the standard form

\[
\begin{pmatrix}
  m & S \\
  S & I_\alpha
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \ddot{\alpha}
\end{pmatrix} + \begin{pmatrix}
  K_h & 0 \\
  0 & K_\alpha
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix} = \begin{pmatrix}
  -l(t) \\
  m(t)
\end{pmatrix}
\]

• Coupled with Theodorsen aerodynamic forces of the form

\[
l(t) = \left\{ \rho \pi b^2 U j \omega \alpha_0 + \pi \rho U c C(k) \left( U \alpha_0 + j \omega h_0 + \left( \frac{3}{4} c - x_f \right) j \omega \alpha_0 \right) + \rho \pi b^2 \left( -\omega^2 h_0 + \left( x_f - \frac{c}{2} \right) \omega^2 \alpha_0 \right) \right\} e^{j \omega t}
\]

With \( k = \omega b / U \)
The equations of motion have coefficients that depend on the frequency. They are known as time-frequency domain equations.

The complete equations are

\[
\begin{pmatrix}
  m & & \\
  S & I_a & \\
  & & 1
\end{pmatrix}
\begin{pmatrix}
  \ddot{h} \\
  \dot{\alpha} \\
  \dot{\alpha}
\end{pmatrix} +
\begin{pmatrix}
  K_h & 0 & \\
  0 & K_a & \\
  & & 1
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha \\
  \alpha
\end{pmatrix} - \frac{1}{2} \rho U^2
\begin{pmatrix}
  -4\pi C(k)jk + 2\pi k^2 & -2\pi cC(k) - 2\pi bjk - 4\pi C(k)\left(\frac{3}{4}c - x_f\right)_j k - 2\pi b^2 k^2 \\
  4\pi ecC(k)jk - 2\pi\left(x_f - \frac{c}{2}\right)k^2 & 2\pi ec^2C(k) - 2\left(\frac{3}{4}c - x_f\right)_j \pi bjk \\
  +4\pi ecC(k)\left(\frac{3}{4}c - x_f\right)_j k + 2\pi\left(x_f - \frac{c}{2}\right)^2 k^2 + \pi \frac{b^2}{4} k^2
\end{pmatrix}
\begin{pmatrix}
  h \\
  \alpha
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0
\end{pmatrix}
\]
Basics

• Solving time-frequency equations of motion is a nonlinear eigenvalue problem.

• The basis of the p-k method is to define

\[ p \equiv \frac{d}{dt} \]

so that

\[ \ddot{q} = p^2 q \]

• We then re-write the equations of motion as

\[ \left( p^2 M_s + K_s - \frac{1}{2} \rho U^2 Q(k) \right) q = 0 \]
Eigenvalue estimation

• These equations of motion are clearly an eigenvalue problem. For a non-trivial solution:

\[ p^2 I + M_s^{-1} \left( K_s - \frac{1}{2} \rho U^2 Q(k) \right) = 0 \]  
\[(1)\]

• This is one equation with two unknowns, we need a second condition.

• Note that \( p^2 \) is an eigenvalue of the matrix

\[ - M_s^{-1} \left( K_s - \frac{1}{2} \rho U^2 Q(k) \right) \]
Eigenvalue estimation (2)

• But $k$ is a frequency: $k = \omega b / U$.
• Recall that the imaginary part of the eigenvalue is the frequency.
• Therefore, our second equation is:

  $$\text{Im}(p) = k \frac{U}{b}$$  \hspace{1cm} (2)

• The problem consists of finding an eigenvalue and a frequency that satisfy both equations (1) and (2).
The p-k solution

• The solution of these equations is iterative.
• We guess a value for the frequency $k$ and then we calculate $p$ from the resulting eigenvalue problem (1).
• The values of $p$ and $k$ should satisfy equation (2).
• If they do not, we change the value of $k$ and re-calculate $p$ until equation (2) is satisfied.
• This is called frequency matching.
Frequency matching

1. Choose an airspeed, $U$.

2. Choose the $i$th degree of freedom ($i = 1$ denotes plunge, $h$, $i = 2$ denotes pitch, $\alpha$).

3. Set an initial value for $\omega$ and, hence, $k$. This value can be $\sqrt{K_h/m}$ if $i = 1$ or $\sqrt{K_\alpha/I_\alpha}$ if $i = 2$.

4. Calculate the eigenvalues $p$ of equation 1.

5. Sort the eigenvalues in ascending order of imaginary part.

6. Choose a new value of $\omega$ equal to the imaginary part of the $i$th sorted eigenvalue and repeat from step 4.

7. When the value of $\omega$ converges stop the iteration. Store the converged value of the $i$th eigenvalue.

8. Repeat from step 2 for all the $i$ degrees of freedom.

9. Repeat from step 1 for other airspeeds
p-k method characteristics

• Converges very quickly to the correct eigenvalue.
• Suitable for large computational problems.
• Calculates complete eigenvalues and therefore damping ratios and natural frequencies.
• Valid at all airspeeds, not just the flutter speed.
• Estimated flutter speeds are very similar to those obtained from the flutter determinant solution.
Sample result

NACA0012 airfoil, y_f=0.4c

Damping Ratio

Airspeed (m/s)

Wagner Function
p-k Method
Roger’s Approximation

• Another way to solve the p-k equations is to transform them completely to the time domain using Roger’s Approximation.

• The frequency-dependent part of equations of motion, \( Q(k) \), is approximated as:

\[
Q(k) = A_0 + A_1j k + A_2(j k)^2 + \sum_{n=1}^{n_l} A_{2+n} \frac{j k}{j k + \gamma_n}
\]  

(3)

• Where \( n_l \) is the number of aerodynamic lags and \( \gamma_n \) are aerodynamic lag coefficients.
Roger’s EOMs

• The equations of motion of the complete aeroelastic system then become:

\[
\dot{\mathbf{q}} = \begin{pmatrix}
-M^{-1}\bar{C} & -M^{-1}\bar{K} & -M^{-1}\bar{A}_3 & \ldots & -M^{-1}\bar{A}_{n_l+2} \\
I & 0 & 0 & \ldots & 0 \\
0 & I & -V\gamma_1/bI & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & I & 0 & \ldots & -V\gamma_{n_l}/bI
\end{pmatrix} \mathbf{q}
\]

• Where

\[
\bar{M} = M_s - \frac{1}{2} \rho b^2 A_2, \quad \bar{C} = C_s - \frac{1}{2} \rho U b A_1, \quad \bar{K} = K_s - \frac{1}{2} \rho U^2 A_0, \quad \bar{A}_j = -\frac{1}{2} \rho U^2 A_j
\]

• Usually:

\[
n_l = 4, \quad \gamma_n = -1.7 k_{\text{max}} \frac{n}{(n_l + 1)^2}, \quad k_{\text{max}} = \text{maximum } k \text{ of interest}
\]
Roger details

• The matrices $A_0$, $A_1$ etc are obtained from equation (3) by performing a least-squares curve fit of $Q(k)$ for several values of $k$.

• If the aeroelastic system has $n$ degrees of freedom, Roger’s equations of motion will have $(n_l+2)n$ states. Of those, $n_l n$ states are aerodynamic states.

• There are similar but more efficient schemes for improving the curve fit of $Q(k)$ and reducing $n_l$, e.g. the minimum state method.
For an aircraft, the matrix $Q(k)$ is obtained using a panel method-based aerodynamic model.

The modelling is usually performed by means of commercial packages, such as MSC.Nastran or Z-Aero.

For a chosen set of $k$ values, e.g. $k_1, k_2, \ldots, k_m$, the corresponding $Q$ matrices are returned.

The $Q$ matrices are then used in conjunction with the $p$-$k$ method to obtain the flutter solution or time-domain responses.

The values of $Q$ at intermediate $k$ values are obtained by interpolation.
Wings are 3D

- All the methods described until now concern 2D wing sections
- These results must now be extended to 3D wings because all wings are 3D
- There are two methods for 3D wing aeroelasticity:
  - Strip theory
  - Panel methods
Strip theory

- Strip theory breaks the wing into spanwise small strips.
- The instantaneous lift and moment acting on each strip are given by the 2D sectional lift and moment theories (quasi-steady, unsteady etc).
Panel methods

- The wing is replaced by its camber surface.
- The surface itself is replaced by panels of mathematical singularities, solutions of Laplace’s equation.
Hancock Model

• A simple 3D wing model is used to introduce 3D aeroelasticity

A rigid flat plate of span $s$, chord $c$ and thickness $t$, suspended through an axis $x_f$ by two torsional springs, one in roll ($K_\gamma$) and one in pitch ($K_\theta$).

The wing has two degrees of freedom, roll ($\gamma$) and pitch ($\theta$).
Equations of motion

• As with the 2D pitch plunge wing, the equations of motion are derived using energy considerations.

• The kinetic energy of a small mass element $dm$ of the wing is given by

$$dT = \frac{1}{2} \dot{z}^2 dm = \frac{1}{2} dm \left( y \dot{y} + (x - x_f) \dot{\theta} \right)^2$$

• The total kinetic energy of the wing is:

$$T = \frac{m}{12} \left( 2s^2 \dot{y}^2 + 3s(c - 2x_f) \dot{y} \dot{\theta} + 2(c^2 - 3x_f c + 3x_f^2) \dot{\theta}^2 \right)$$
Structural equations

• The potential energy of the wing is simply

\[ V = \frac{1}{2} K_\gamma \gamma^2 + \frac{1}{2} K_\theta \theta^2 \]

• The full structural equations of motion are then:

\[
\begin{pmatrix}
I_\gamma & I_{\gamma\theta} \\
I_{\gamma\theta} & I_\theta
\end{pmatrix}
\begin{bmatrix}
\ddot{\gamma} \\
\ddot{\theta}
\end{bmatrix}
+ \begin{pmatrix}
K_\gamma & 0 \\
0 & K_\theta
\end{pmatrix}
\begin{bmatrix}
\gamma \\
\theta
\end{bmatrix}
= \begin{bmatrix}
M_1 \\
M_2
\end{bmatrix}
\]

\[ I_\gamma = \frac{ms^2}{3}, \quad I_{\gamma\theta} = m\left(c - 2x_f\right)s/4, \quad I_\theta = m\left(c^2 - 3x_f c + 3x_f^2\right)/3 \]
Strip theory

• The quasi-steady or unsteady approximations for the lift and moment around the flexural axis are applied to infinitesimal strips of wing

• The lift and moment on these strips are integrated over the entire span of the wing

• The result is a quasi-steady pseudo-3D lift and moment acting on the Hancock wing

\[ M_1 = - \int_0^s y l(y) \, dy \]

\[ M_2 = - \int_0^s m_x f(y) \, dy \]
Quasi-steady strip theory

• Denote, $\theta=\alpha$ and $h=y\gamma$. Then

$$l = \left\{ \rho \pi b^2 \left( y\ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \dot{\theta} \right) + \rho \pi b^2 U \dot{\theta} + \rho U^2 c \pi \left( \theta(t) + \frac{y\dot{\gamma}(t)}{U} + \left( \frac{3}{4} c - x_f \right) \frac{\dot{\theta}(t)}{U} \right) \right\}$$

$$m_{xf} = \left\{ \rho \pi b^2 \left( x_f - \frac{c}{2} \right) \left( y\ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \dot{\theta} \right) - \frac{\rho \pi b^4}{8} \ddot{\theta} - \left( \frac{3}{4} c - x_f \right) \rho \pi b^2 U \dot{\theta} + \rho U^2 ec^2 \pi \left( \theta(t) + \frac{y\dot{\gamma}(t)}{U} + \left( \frac{3}{4} c - x_f \right) \frac{\dot{\theta}(t)}{U} \right) - \frac{1}{16} \rho U^3 e \pi \dot{\theta} \right\}$$

• Carrying out the strip theory integrations will yield the total moments around the $y=0$ and $x=x_f$ axes.
3D Quasi-steady equations of motion

• The full 3D quasi-steady equations of motion are given by

\[
\begin{pmatrix}
I_\gamma & I_\gamma \theta \\
I_\gamma \theta & I_\theta
\end{pmatrix} + \rho \pi b^2 s \begin{pmatrix}
s^2/3 \\
-(x_f - c/2)s/2 & (x_f - c/2)^2 + b^2/8
\end{pmatrix} \begin{Bmatrix}
\dot{\gamma} \\
\dot{\theta}
\end{Bmatrix} +
\rho U c \pi s \begin{pmatrix}
s^2/3 & \frac{1}{2} s \left( \left( \frac{3}{4} c - x_f \right) + \frac{c}{4} \right) \\
-e c s/2 & \left( \frac{3}{4} c - x_f \right) \left( \frac{c}{2} - x_f \right) + \frac{c^2}{16}
\end{pmatrix} \begin{Bmatrix}
\ddot{\gamma} \\
\ddot{\theta}
\end{Bmatrix} +
\begin{pmatrix}
K_\gamma & 1/2 \rho U^2 c \pi s^2 \\
0 & K_\theta - \rho U^2 e c^2 \pi s
\end{pmatrix} \begin{Bmatrix}
\gamma \\
\theta
\end{Bmatrix} = \begin{Bmatrix}
0 \\
0
\end{Bmatrix}
\]

• They can be solved as usual
Natural frequencies and damping ratios
Theodorsen function aerodynamics

• Again, Theodorsen function aerodynamics (unsteady frequency domain) can be implemented directly using strip theory:

\[
\begin{align*}
L & = \left\{ \rho \pi U c C(k) \left( U \theta + y \dot{\gamma} + \left( \frac{3}{4} c - x_f \right) \dot{\theta} \right) + \rho \pi b^2 \dot{\theta} + \\
& \quad \rho \pi b^2 \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \ddot{\theta} \right) \right\} \\
m_{xf} & = \left\{ \rho \pi U e c^2 C(k) \left( U \theta + y \dot{\gamma} + \left( \frac{3}{4} c - x_f \right) \dot{\theta} \right) - \left( \frac{3}{4} c - x_f \right) \rho \pi b^2 \dot{\theta} + \\
& \quad \rho \pi b^2 \left( x_f - \frac{c}{2} \right) \left( y \ddot{\gamma} - \left( x_f - \frac{c}{2} \right) \ddot{\theta} \right) - \rho \pi \frac{b^4}{8} \ddot{\theta} \right\}
\end{align*}
\]
Flutter determinant

- The flutter determinant for the Hancock model is given by

\[
Q = \begin{pmatrix}
K_\gamma + \frac{1}{3}\omega \pi \rho s^2 (jUeC(k) - \omega b^2) & \frac{1}{2}\pi \rho s^2 \left\{ U^2 eC(k) \left(1 + j\omega \left( \frac{3}{4}c - x_f \right) \right) + \omega b^2 \left(U j + \omega \left(x_f - \frac{c}{2}\right)\right) \right\} \\
\frac{1}{2}\omega \pi \rho s^2 \left(-jU e^2 C(k) + b^2 \left(x_f - \frac{c}{2}\right)\right) & \pi \rho s \left\{ -U^2 eC^2(k) + j\omega U \left( \frac{3}{4}c - x_f \right) (eC^2(k) + b^2) \\
& -\omega^2 \left( \frac{b^4}{8} + b^2 \left(x_f - \frac{c}{2}\right)^2 \right) \right\} + K_\theta
\end{pmatrix}
\]

- And is solved in exactly the same was as for the 2D pitch-plunge model.
Instead of solving the flutter determinant, we can apply the p-k method.

The process is exactly as presented in the 2D case.
Comparison of flutter speeds

Wagner and Theodorsen solutions are identical.

Quasi-steady solution is the most conservative

Ignore the other solutions
Comparison of 3D and strip theory, static case

The 3D lift distribution is completely different to the strip theory result!

Strip theory:

\[ C_L = 2\pi \alpha \]

Lifting line method:

\[ C_L = \frac{2\pi AR}{AR + 2} \alpha \]
Vortex lattice aerodynamics

- Strip theory is a very gross approximation that is only exact when the wing’s aspect ratio is infinite.
- It is approximately correct when the aspect ratio is very large.
- It becomes completely unsatisfactory at moderate and small aspect ratios (less than 8-10).
Vortex lattice method

- The basis of the VLM is the division of the wing planform into panels on which lie vortex rings, usually called a vortex lattice.
- A vortex ring is a rectangle made up of four straight line vortex segments.
- It is an elementary solution of Laplace’s equation.
Characteristics of a vortex ring

• The vector $n_{ij}$ is a unit vector normal to the ring and positioned at its midpoint (the intersection of the two diagonals - also termed the collocation point)

• The vorticity, $\Gamma$, is constant over all the 4 segments of a vortex ring.

• Each segment is inducing a velocity $[u \ v \ w]$ at a general point $P$.

• In the case where the point $P$ lies on a vortex ring segment, the velocity induced is 0.
Panelling up and solving

- The process starts by dividing a wing planform into panels.
- The wing can be swept, tapered and twisted.
- It cannot have thickness; the panels lie on the wing’s camber surface.
- The wake must also be panelled up.
- The object of the VLM is to calculate the values of the vorticities $\Gamma$ on each wing panel at each instance in time.
- The vorticity of the wake panels does not change in time. Only the vorticities of the wing panels are unknowns.
Boundary conditions

• The vorticities $\Gamma$ on each wing panel at each instance in time are calculated by enforcing boundary conditions:
  – Impermeability: there is no flow normal to the panel at its collocation point. This condition gives all the $\Gamma$ values.
  – Kutta condition: the flow must separate at the trailing edge. This condition is satisfied automatically by placing the leading edge of each vortex ring on the quarter-chord of its corresponding geometric panel.
Calculating forces

• Once the vorticities on the wing panels are known, the lift and moment acting on the wing can be calculated.

• These are calculated from the pressure difference acting on each panel.

• Summing the pressure differences of the entire wing yields the total forces and moments.
Panels for static wing

Even if the wing is not moving, the wake must still be modelled because it describes the downwash induced on the wing’s surface and, hence, the induced drag.
Wake shapes

Wake shape behind a rectangular wing that underwent an impulsive start from rest.

The aspect ratio of the wing is 4 and the angle of attack 5 degrees.
Effect of Aspect Ratio on lift coefficient

Lift coefficient variation with time for an impulsively started rectangular wing of varying Aspect Ratio. It can be clearly seen that the 3D results approach Wagner’s function (2D result) as the Aspect Ratio increases.
Free wakes

- Free wake behind a flapping goose wing.
- Airspeed: 18m/s.
Industrial use

• Unsteady wakes are beautiful but expensive to calculate.
• For practical purposes, a fixed wake is used with unsteady vorticity, just like Theodorsen’s method.
• The wake propagates at the free stream airspeed and in the free stream direction.
• Only a short length of the wake is simulated (a few chord-lengths).
• The result is a linearized aerodynamic model.
Aerodynamic force coefficient matrices

• 3D aerodynamic calculations can be further speeded up by calculating everything in terms of the mode shapes of the structure.

• This treatment allows the expression of the aerodynamic forces as modal aerodynamic forces, written in terms of aerodynamic force coefficients matrix $Q(k)$.

• These are square matrices with dimensions equal to the number of retained modes. They also depend on response frequency.

• Therefore, the complete aeroelastic system can be written as a set of linear ODEs with frequency-dependent matrices, to be solved using the p-k method.
Case study

- The plate is still not perfectly symmetric but, as it is swept, no static deformations appear.
- The oscillations start at 16.8 m/s. They combine bending and torsion at the same frequency.
- At the highest airspeeds the motion becomes slightly irregular in bending.
  - On balance, the oscillations are less complex than in the zero sweep case.

$U=0 \text{ m/s}$

$U=21 \text{ m/s}$
Mode shapes

Mode 1

Mode 2

Mode 3
VLM calculation

- The wing is always flat and never deforms.
- The wake propagates with the free stream and never deforms.
- The downwash on the panels depends on the free stream, angle of attack and modal out-of-plane displacements.
- Only the lift does work on the structure.
- The lift equation is Fourier Transformed and a frequency-dependent generalized force matrix is developed.
- The number of chordwise and spanwise panels is 30 each.
Applying impermeability boundary condition:

\[ Q_\infty \text{diag} \left( \hat{U}n^T \right) - Uw_x r(t) - wi(t) + A_b \Gamma_b(t) + A_w \Gamma_w(t) = 0 \]

where

\[ \Gamma_w(t) = \begin{pmatrix} P_c \Gamma_b(t - \Delta t) \\ P_c \Gamma_b(t - 2\Delta t) \\ \vdots \\ P_c \Gamma_b(t - m_w \Delta t) \end{pmatrix} \]

Transforming to the frequency domain:

\[ \Gamma_w(\omega) = P_e(\omega) P_c \Gamma_b(\omega) \]

where

\[ P_e(\omega) = \begin{pmatrix} I_n e^{-i\omega \Delta t} \\ I_n e^{-i\omega 2\Delta t} \\ \vdots \\ I_n e^{-i\omega m_w \Delta t} \end{pmatrix} \]

So that, finally

\[ \Gamma_b(\omega) = - \left( A_b + A_w P_e(\omega) P_c \right)^{-1} \left( \text{diag} \left( Q_\infty \hat{U}n^T \right) \delta(\omega) - (Uw_x + i\omega w) r(\omega) \right) \]
VLM maths (2)

• This means that, in the frequency domain, we can calculate the bound vorticity exclusively as a function of the modal downwash and free stream.

• The linearized lift is a multiple of the bound vorticity: \( L(\omega) = \rho (Q_\infty G_{cs} + i\omega G_A) \Gamma_b(\omega) \)

• So that we can now calculate the work done by the lift, i.e. the generalized aerodynamic force matrix:

\[
Q(k) = -\rho Q^2_\infty (Q_0(0) - Q_1(k)r(k))
\]
Flutter solution

- $10^\circ$ sweep, $\text{AR}=4$, 5 structural modes
- Flutter airspeed 15.8 m/s, flutter frequency 5.3 Hz.
Flutter speed predictions

- Comparison of simulated and experimental flutter airspeeds and frequencies for all AR=4 wings.
- Linear structure, p-k solution.

![Graph of Flutter Speed vs Sweep Angle](image1)
![Graph of Flutter Frequency vs Sweep Angle](image2)
Commercial packages

• There are two major commercial packages that can calculate 3D unsteady aerodynamics using panel methods:
  – MSC.Nastran (MSC Software)
  – ZAERO (ZONA Technology)

• They can both deal with complex aircraft geometries.
ZAERO AFA example

• The examples manual of ZAERO features an Advanced Fighter Aircraft model.
BAH Example

- Bisplinghoff, Ashley and Halfman wing

- FEM with 12 nodes and 72 dof
First 5 modes of BAH wing
GTA Example

Here is a very simple aeroelastic model for a Generic Transport Aircraft

Finite element model: Bar elements with 678 degrees of freedom

Aerodynamic model: 2500 doublet lattice panels
First 7 flexible modes. Clear flutter mechanism between first and third mode (first wing bending and aileron deflection)
Time domain plots for the GTA

\[ V < V_F \]

\[ V = V_F \]
Supersonic Transport

• The SST was a proposal for the replacement of the Concorde.

• The aeroelastic model is a half-model.

• The aerodynamics features the wing and a rectangle for the wall.
Flutter plots for SST

First 9 flexible modes. Clear flutter mechanism between first and third mode.
Aeroelastic design in industry is almost exclusively carried out using lattice methods (vortex or doublet) combined with a Finite Element model with few retained modes (generally fewer than 100).

Panel methods (source and doublet) are also used for more detailed geometric representation, including thickness.

The aerodynamic forces are written in the form of Aerodynamic Force Coefficient Matrices that depend on frequency.

The flutter solution is obtained using the p-k method.