Aerothermodynamics of high speed flows

Lecture 4: Flow with discontinuities, oblique shocks

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Outline

1. Oblique shocks
2. Shock reflection and shock interaction
3. Prandtl-Meyer expansion
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Oblique shocks

X-15 model fired into a wind Tunnel at M= 3.5 [NASA]

Mach waves in a supersonic nozzle [Meyer, 1908]
The bow shock wave on the bullet, in passing over the holes in the plate, sends out weak disturbances above the place which coalesce into a Mach wave above the plate [Stanford U]
Supersonic bullet passing under a perforated plate

The bow shock wave on the bullet, in passing over the holes in the plate, sends out weak disturbances above the place which coalesce into a Mach wave above the plate [Stanford U]
Source of oblique waves

- A source constantly emits sound waves as it moves through a stationary gas at speed $u$

![Diagram of oblique waves](image)

Propagation of disturbances in subsonic and supersonic flow

- When $u < a$, the source always stays inside the family of circular sound waves and the waves continuously move ahead of the source.
- When $u > a$, the source is constantly outside the family of circular sound waves, it is moving ahead of the wave front. The wave fronts form a disturbance envelope called Mach wave.

- The Mach angle $\mu$ is determined as $\sin \mu = \frac{1}{M}$.
Oblique shocks

Steady 2D discontinuity

Rankine-Hugoniot jump relations for steady 2D discontinuity

\[
\begin{align*}
\rho_2 u_{n2} &= \rho_1 u_{n1} \\
\rho_2 u_{n2}^2 + p_2 &= \rho_1 u_{n1}^2 + p_1 \\
\rho_2 u_{n2} u_{t2} &= \rho_1 u_{n1} u_{t1} \\
\rho_2 u_{n2} H_2 &= \rho_1 u_{n1} H_1 
\end{align*}
\]

- \( u_{n1} = u_{n2} = 0 \): contact discontinuity, slip line \( \Rightarrow p_2 = p_1 \)
- \( u_{n1}, u_{n2} \neq 0 \): oblique shock \( \Rightarrow u_{t2} = u_{t1} \) and \( H_2 = H_1 \)

\[
\frac{\rho_2 u_{n2}}{u_{t2}} = \frac{\rho_1 u_{n1}}{u_{t1}} \quad \Rightarrow \quad \tan(\beta - \theta) = \frac{\tan\beta}{\rho_2/\rho_1}
\]
Oblique shocks

The total enthalpy is

\[ H = h + \frac{1}{2} (u_n^2 + u_t^2) = \frac{1}{2} \frac{\gamma+1}{\gamma-1} a^*^2 \]

The jump relations

\[ \rho_2 u_{n2} = \rho_1 u_{n1} \]
\[ \rho_2 u_{n2}^2 + p_2 = \rho_1 u_{n1}^2 + p_1 \]
\[ h_2 + \frac{1}{2} u_{n2}^2 = h_1 + \frac{1}{2} u_{n1}^2 = H - \frac{1}{2} u_t^2 \]

are identical to the normal shock relations

⇒ For a given shock angle \( \beta \) and incoming Mach number \( M_1 \), all flow quantities behind the shock can be directly computed by the 1D relations,

- \[ \frac{\rho_2}{\rho_1} = \frac{u_{n1}}{u_{n2}} = \frac{(\gamma+1)M_{n1}^2}{2+(\gamma-1)M_{n1}^2} \text{ and } \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} (M_{n1}^2 - 1) \]

- \[ \frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right) / \left( \frac{\rho_2}{\rho_1} \right) = \left[ 1 + \frac{2\gamma}{\gamma+1} (M_{n1}^2 - 1) \right] \left[ \frac{2+(\gamma-1)M_{n1}^2}{(\gamma+1)M_{n1}^2} \right] \]

- \[ M_{n2}^2 = \frac{1+\frac{\gamma-1}{\gamma} M_{n1}^2}{\frac{\gamma M_{n1}^2}{\gamma M_{n1}^2 - \gamma-1}} \]

based on normal components \( M_{n1} = M_1 \sin \beta \) and \( M_{n2} = M_2 \sin(\beta - \theta) \)
The $\theta - \beta - M_1$ polar relation derived based on trigonometry:

\[
\tan(\beta - \theta) = \frac{\tan \beta}{\rho_2/\rho_1}
\]

\[
\frac{\tan \beta - \tan \theta}{1 + \tan \beta \tan \theta} = \alpha \tan \beta, \quad \text{with} \quad \alpha = \left[ 2 + (\gamma - 1)M_1^2 \sin^2 \beta \right] \left[ \gamma + 1 \right] M_1^2 \sin^2 \beta
\]

\[
\tan \theta = \left[ \frac{1 - \alpha}{1 + \alpha \tan^2 \beta} \right] \tan \beta
\]

\[
\Rightarrow \quad \tan \theta = \left[ \frac{M_1^2 \sin^2 \beta - 1}{M_1^2 (\gamma + \cos 2\beta) + 2} \right] 2 \cot \beta
\]
Oblique shocks

Magin (AERO 0033–1)  Aerothermodynamics  2016-2017  10 / 27
Maximum deflection angle for oblique shocks

For any given $M_1$, there is a maximum deflection angle $\theta_{\text{max}}$. If the physical geometry is such that $\theta > \theta_{\text{max}}$, then no solution exists for a straight oblique shockwave. Instead the shock will be curved and detached.
Strong shock and weak shock solutions

For any given $\theta < \theta_{\text{max}}$, there are 2 values of $\beta$ predicted.

- Changes across the shock are more severe for the larger value of $\beta$ (strong shock).
- The weak shock is favoured and usually occur unless, for instance, the back pressure is increased by some independent mechanism.
- In the weak shock solution, $M_2 > 1$ except for a small region near $\theta_{\text{max}}$.
- If $\theta = 0$, then $\beta = \pi/2$ (normal shock) or $\beta = \mu$ (Mach wave, evanescent shock).
Detached shock in front of a blunt body

The shape of the detached shock in front of a blunt body wave can be obtained by means of computational fluid dynamics simulations.

- **Evanescent shock** (zero deflection: \( \sin \beta = 1/M_1 \), Mach wave)
- **Weak shock**, \( M_2 > 1 \)
- **Weak shock**, \( M_2 = 1 \)
- **Maximum flow deflection point dividing weak** \((M_2 > 1)\) **and strong shock solutions**
- **Strong shock**
- **Normal shock**
Prandtl’s relation for oblique shocks

- A normal Bernoulli constant $a^{**^2} = a^{*^2} - \frac{\gamma-1}{\gamma+1} u_t^2$ is introduced to express conservation of the normal component of the total enthalpy

$$h_2 + \frac{1}{2} u_{n2}^2 = h_1 + \frac{1}{2} u_{n1}^2 = \frac{1}{2} \frac{\gamma+1}{\gamma-1} a^{**^2}$$

- Prandtl’s relation for oblique shocks is $a^{**^2} = u_{n1} u_{n2}$

- A critical normal Mach number can be defined as $M_n^* = u_n / a^{**}$, and Prandtl’s relation for oblique shocks is expressed as

$$M_{n1}^* M_{n2}^* = 1$$

- Notice that $M_{t1} \neq M_{t2}$ since $a_1 \neq a_2$
- The critical tangential Mach number defined as $M_t^* = u_t / a^{**}$ is constant: $M_{t1}^* = M_{t2}^*$ since $a^{**} = \text{constant}$
Outline

1. Oblique shocks
2. Shock reflection and shock interaction
3. Prandtl-Meyer expansion
Shock polar in hodograph plane (exact oblique shock theory)

- The shock polar in the hodograph plane \((u_2, v_2) = (V_2 \cos \theta, V_2 \sin \theta)\), with \(V_2 = (u_2^2 + v_2^2)^{1/2}\) is derived from Prandtl’s relation:
  \[
  u_{n1} u_{n2} = a^*^2 - \frac{\gamma - 1}{\gamma + 1} u_t^2
  \]

\[
\begin{align*}
  u_{n1} &= u_1 \sin \beta \\
  u_{n2} &= u_{n1} - \frac{v_2}{\cos \beta} = u_1 \sin \beta - \frac{v_2}{\cos \beta} \\
  u_t &= u_1 \cos \beta
\end{align*}
\]

\[
\Rightarrow u_1^2 \sin^2 \beta - u_1 v_2 \tan \beta = a^*^2 - \frac{\gamma - 1}{\gamma + 1} u_1^2 \cos^2 \beta
\]

- Defining non-dimensional velocities \(\bar{u}_2 = u_2 / a^*, \bar{v}_2 = v_2 / a^*, \) and \(\bar{u}_1 = u_1 / a^*\), this relation is expressed as
  \[
  \bar{v}_2^2 = (\bar{u}_2 - \bar{u}_1)^2 \frac{\bar{u}_1 \bar{u}_2 - 1}{1 + \frac{2}{\gamma + 1} \bar{u}_2^2 - \bar{u}_1 \bar{u}_2}
  \]
Proof

Considering the relation \( \tan \beta = \frac{u_1 - u_2}{v_2} \), obtained based on the grey triangle, and the trigonometric relations \( \cos^2 \beta = \frac{1}{1 + \tan^2 \beta} \) and \( \sin^2 \beta = \frac{\tan^2 \beta}{1 + \tan^2 \beta} \), one gets

\[
u_1^2 \frac{(u_1 - u_2)^2}{v_2^2 + (u_1 - u_2)^2} - u_1(u_1 - u_2) = a^*2 - \frac{\gamma - 1}{\gamma + 1} \frac{u_1^2 v_2^2}{v_2^2 + (u_1 - u_2)^2} \]

Dividing by \( a^*2 \)

\[ar{u}_1^2(\bar{u}_1 - \bar{u}_2^2) - \bar{u}_1(\bar{u}_1 - \bar{u}_2)[\bar{v}_2^2 + (\bar{u}_1 - \bar{u}_2)^2] = \bar{v}_2^2 + (\bar{u}_1 - \bar{u}_2)^2 - \frac{\gamma - 1}{\gamma + 1} \bar{u}_1 \bar{v}_2^2 \]

After some algebra

\[ar{v}_2^2(1 + \frac{2}{\gamma + 1} \bar{u}_2^2 - \bar{u}_1 \bar{u}_2) = (\bar{u}_2 - \bar{u}_1)^2(\bar{u}_1 \bar{u}_2 - 1) \]

No deflection \( (\theta = 0, \bar{v}_2 = 0) \)

- \( u_2 = u_1 \): evanescent shock (Mach line)
- \( \bar{u}_2 = 1/\bar{u}_1 \): normal shock (Prandtl relation)

Deflection angle \( 0 < \theta < \theta_{max} \): 2 values for \( v_2 \)

- Strong shock solution (always supersonic flow)
- Weak shock solution (either subsonic or supersonic flow)

Maximum deflection angle \( \theta = \theta_{max} \)

Deflection angle \( \theta > \theta_{max} \): no solution (detached shock)
For $M_1 = 2$

- $\theta = 0^\circ$ Point A: evanescent shock, Point F: normal shock
- $0 < \theta = 10^\circ < \theta_{\text{max}}$, Point B: weak shock, Point E: strong shock
- $\theta = \theta_{\text{max}}$, Point D: maximum deflection angle
- Point C: separation between supersonic and subsonic regimes (critical Mach number $M^*$ behaves as the local Mach number $M$)
Shock polar in pressure-deflection plane

- Pressure-deflection diagram: locus of all possible static pressure behind an oblique shock wave as a function of the deflection angle

![Pressure-deflection diagram](image)

- It is a parametric curve depending on the free stream Mach number $M_1$: \[ \frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 \sin^2 \beta - 1), \text{ where } \beta = \beta(M_1, \theta) \]

- Points B and B’: weak shock solutions depending on the sign of $\theta$
- Points E and E’: strong shock solutions
- Points G and G’: expansion shock (not physical)
Shock reflection from a solid boundary

- Consider an **oblique shock** wave deviating the flow from region 1 to 2 through an angle $-\Delta\theta$ incident on a solid wall at point 1

- The streamline is deflected from region 2 to 3 through a **reflected shock** to leave the flow parallel to the wall
  - $M_1 = 2$ and $\Delta\theta = -10^\circ$
    - $M_2 = 1.641$ and $p_2/p_1 = 1.707$
  - $M_3 = 1.641$ and $\Delta\theta = +10^\circ$
    - $M_2 = 1.285$ and $p_3/p_1 = 2.805$

- In the pressure-deflection plane
  - $M_2 < M_1$: the reflected shock is weaker than the oblique shock
  - $p_3 - p_2 > p_2 - p_1$
  - The shock is not specularly reflected
Approximate solution based on the characteristic theory

- On $s^+$ characteristic between regions 1 and 2

$$\nu(M_2) = \nu(M_1) + \Delta \theta = 26.379^\circ - 10^\circ = 16.379^\circ \Rightarrow M_2 = 1.651$$

- Isentropic relation

$$\frac{p_2}{p_1} = \left(\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2}\right)^{\frac{\gamma}{\gamma - 1}} = 1.705$$

- On $s^-$ characteristic between regions 2 and 3

$$\nu(M_3) = \nu(M_2) - \Delta \theta = 16.379^\circ - 10^\circ = 6.379^\circ \Rightarrow M_3 = 1.308$$

- Isentropic relation

$$\frac{p_3}{p_1} = \left(\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_3^2}\right)^{\frac{\gamma}{\gamma - 1}} = 2.795$$
Singular shock reflection from a solid boundary

- When the deflection $\Delta \theta$ is larger than the maximum deflection angle $\theta_{max}$ for the $M_2$ value, no regular shock reflection is allowed
  - $M_1 = 2$ and $\Delta \theta = -16^\circ$
  - $M_2 = 1.403$ and $p_2/p_1 = 2.308$
  - $M_2 = 1.403$ and $\Delta \theta = +16^\circ$
  - no oblique shock solution

- A normal shock is formed at the wall to allow the streamlines to continue parallel to the wall (Mach stem)

- Away from the wall, this normal shock transits into a curved shock which intersects the incident shock
  - Singular reflexion: the reflected shock is curved in reality
  - Slip line $\Sigma$ between regions 3 and 4 results from different entropy jump
  - Intersection between two polars: $p_3 = p_4 (< p_5) \Rightarrow \Delta \theta < 16^\circ$
  - Subsonic region: no analytical result

- **Triple point** $T$: incident & reflected shocks + Mach stem (+slip line)
Shock interaction

- Consider two **oblique shocks** of the same intensity deviating the flow from region 1 to 2 through an angle $|\Delta \theta|$ and interacting at point I.

- The streamlines are deflected from region 2 to 3 through **refracted shocks** to leave the flow parallel to the symmetry plane.

  - $M_1 = 1.95$ and $\Delta \theta = -10^\circ$
  - $\Rightarrow M_2 = 1.594$ and $p_2/p_1 = 1.694$
  - $M_2 = 1.594$ and $\Delta \theta = +10^\circ$
  - $\Rightarrow M_3 = 1.233$ and $p_3/p_1 = 2.783$

- In the pressure-deflection plane:
  - $M_2 < M_1$: the refracted shock is weaker than the oblique shock
  - $p_3 - p_2 > p_2 - p_1$
  - No slip line due to the symmetry of the problem.
Approximate solution based on the characteristic theory

- On $s^+$ characteristic between regions 1 and 2
  \[ \nu(M_2) = \nu(M_1) + \Delta \theta = 24.992^\circ - 10^\circ = 14.992^\circ \Rightarrow M_2 = 1.604 \]

- On $s^-$ characteristic between regions 2 and 3
  \[ \nu(M_3) = \nu(M_2) - \Delta \theta = 14.992^\circ - 10^\circ = 4.992^\circ \Rightarrow M_3 = 1.256 \]
**Singular shock interaction** (∼ type II interference, Edney, 1968)

- When the deflection $\Delta \theta$ is larger than the maximum deflection angle $\theta_{\text{max}}$ for the $M_2$ value, no regular shock refraction is allowed.
  - $M_1 = 1.95$ and $\Delta \theta = -14^\circ$
  - $M_2 = 1.440$ and $p_2/p_1 = 2.069$
  - $M_2 = 1.440$ and $\Delta \theta = +14^\circ$
  - $\Rightarrow$ no oblique shock solution

- A normal shock is formed between the two incidents shocks to allow the streamlines to continue parallel to the symmetry plane (**Mach disk** for revolution flow)

- Away from the wall, this normal shock transits into a curved shock which intersects the incident shock
  - Two **triple points** $T_1$ and $T_2$
  - Two **slip lines** $\Sigma_1$ and $\Sigma_2$ between regions 3 and 4 result from different entropy jump
  - Intersection between two polars: $p_3 = p_4 \Rightarrow \Delta \theta < 14^\circ$
Outline

1. Oblique shocks
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When a supersonic flow is turned away from itself, an expansion wave is formed.

- $M_2 > M_1$ an expansion corner is a means to increase the Mach number
- $p_2 < p_1$, $\rho_2 < \rho_1$, $T_2 < T_1$
- The expansion fan is a continuous expansion region composed of an infinite number of Mach waves and bounded by two Mach waves
- The expansion is isentropic
The Mach number of a Mach wave is given by

$$\nu(M) = \nu(M_1) - \theta = \nu(M_1) + |\theta|$$

where $\theta$ is the flow angle (see characteristics theory in lecture 2).

- The Prandtl-Meyer function is an increasing function of the Mach number: 
  $$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \sqrt{\frac{\gamma-1}{\gamma+1} (M^2 - 1)} - \arctan \sqrt{M^2 - 1}$$

- In particular, the flow in region 2 is given by 
  $$\nu(M_2) = \nu(M_1) + |\theta_2|$$
The maximum turning angle is obtained when the end Mach number approaches infinity. In this case, Prandtl-Meyer function becomes

$$\nu(M \to \infty) = \frac{\pi}{2} \left( \sqrt{\frac{\gamma + 1}{\gamma - 1}} - 1 \right)$$

The maximum turning angle = $$\nu(M \to \infty) - \nu(M_1)$$

When the deflection angle exceeds the maximum angle, the flow in this case behaves as if there is almost a maximum angle and in that region beyond the flow will become a vortex street.