Aerothermodynamics of high speed flows

AERO 0033–1

Lecture 6: 2D potential flow, method of characteristics

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Outline

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Entropy eq. (Navier-Stokes)

- First law of thermodynamics for open system: $T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}(\frac{1}{\rho})$
- Mass and thermal energy (Navier-Stokes) eqs.

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(e) = -p\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{q} - \boldsymbol{\tau} : \nabla\boldsymbol{u}$$
$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho) = -\rho\nabla\cdot\boldsymbol{u}$$

$\Rightarrow$ Entropy eq. $\rho\frac{\mathrm{D}}{\mathrm{D}t}(s) = \frac{\rho}{T}\frac{\mathrm{D}}{\mathrm{D}t}(e) - \frac{p}{\rho T}\frac{\mathrm{D}}{\mathrm{D}t}(\rho) = -\frac{1}{T}\nabla\cdot\boldsymbol{q} - \frac{1}{T}\boldsymbol{\tau} : \nabla\boldsymbol{u}$

- Transport fluxes: $\boldsymbol{\tau} = -2\eta\mathbf{S}$ and $\boldsymbol{q} = -\lambda\nabla T$
- Conservative form: $\partial_t(\rho s) + \nabla\cdot(\rho\boldsymbol{u}s) + \nabla\cdot(\frac{\boldsymbol{q}}{T}) = \Upsilon$
- The entropy production rate is nonnegative

$$\Upsilon = \frac{\eta}{T}\mathbf{S}{:}\mathbf{S} + \frac{\lambda}{T^2}|\nabla T|^2 \geq 0$$

- In smooth inviscid flows, the entropy is constant along a trajectory (pathline)

$$\frac{\mathrm{D}}{\mathrm{D}t}(s) = 0$$

Total enthalpy eq.

- Total energy $E = e + \frac{1}{2}|\boldsymbol{u}|^2$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(E) + \nabla\cdot(p\boldsymbol{u}) + \nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\tau}\cdot\boldsymbol{u}) = 0$$

- Total enthalpy $H = h + \frac{1}{2}|\boldsymbol{u}|^2 = E + p/\rho$
  - $\rho\frac{\mathrm{D}}{\mathrm{D}t}(\frac{p}{\rho}) = \frac{\mathrm{D}}{\mathrm{D}t}(p) - \frac{p}{\rho}\frac{\mathrm{D}}{\mathrm{D}t}(\rho) = \partial_t p + \boldsymbol{u}\cdot\nabla p + p\nabla\cdot\boldsymbol{u}$
    $= \partial_t p + \nabla\cdot(p\boldsymbol{u})$

$\Rightarrow$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(H) - \partial_t p + \nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\tau}\cdot\boldsymbol{u}) = 0$$

- In steady inviscid flows, the total enthalpy is constant along a trajectory (pathline)

$$\frac{\mathrm{D}}{\mathrm{D}t}(H) = 0$$

Crocco's eq.

- Momentum eq.: $\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}$
  - Lagrange identity: $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u})$
  - Vorticity vector: $\omega = \nabla\times\boldsymbol{u}$
  - First law of thermodynamics: $\mathrm{d}h = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p \Rightarrow \frac{1}{\rho}\nabla p = \nabla h - T\nabla s$

  $\Rightarrow \partial_t \boldsymbol{u} + \omega\times\boldsymbol{u} + \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) = T\nabla s - \nabla h - \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}$
- Crocco's eq

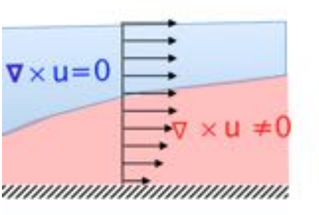
$$\partial_t \boldsymbol{u} + \omega\times\boldsymbol{u} = T\nabla s - \nabla H - \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}$$

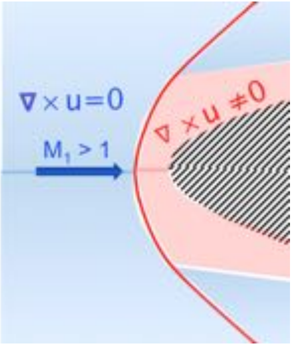
- In steady inviscid flows, with constant H ($\nabla H = 0$), the entropy is constant in irrotational flows ($\omega = 0$)

$$\omega\times\boldsymbol{u} = T\nabla s$$

  Conversely, if the flow is isentropic $\nabla s = 0$, it must be irrotational unless $\omega \parallel \boldsymbol{u}$ everywhere (Beltrami flow)
- In smooth inviscid and steady flows, the entropy vanishes on a pathline: $\boldsymbol{u}\cdot\nabla s = 0$. This property does not imply that $\nabla s = 0$

Rotational flows





Exemples of rotational flow: boundary layer – detached shock

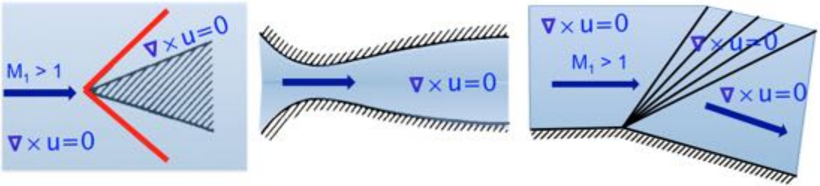
$$\omega = \nabla \times \boldsymbol{u} \neq 0 \Rightarrow \nabla s \neq 0$$

Irrotational flows

- Steady flow
- Uniform upstream conditions
- Isentropic flow
  - Inviscid flow (no dissipation)
  - No discontinuities

$\Rightarrow$ Potential (irrotational) flow

- $\omega = \nabla \times \boldsymbol{u} = 0$
- $\boldsymbol{u} = \nabla \phi$



Exemples of irrotational flow: attached shock (except shock region), nozzle, expansion

Alternative form of continuity eq.

- Continuity eq.: $\frac{\mathrm{D}}{\mathrm{D}t}(\rho) + \rho\nabla\cdot\boldsymbol{u} = 0$
- Let us consider a smooth inviscid and steady flow
  - The material derivative reads $\frac{\mathrm{D}}{\mathrm{D}t}(\rho) = \boldsymbol{u}\cdot\nabla\rho$
  - The entropy is conserved on a path line: $\boldsymbol{u}\cdot\nabla s = 0$
  - By definition of the speed of sound: $\boldsymbol{u}\cdot\nabla\rho = \frac{1}{a^2}\boldsymbol{u}\cdot\nabla p$
  - Multiplying by $\boldsymbol{u}$ the momentum eq.:

$$\boldsymbol{u}\cdot\nabla p = -\boldsymbol{u}\cdot(\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\rho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{u}$$

$$\Rightarrow \frac{\mathrm{D}}{\mathrm{D}t}(\rho) = -\frac{\rho}{a^2}\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{u}$$

- Alternative form of the continuity eq. valid for **both rotational and irrotational** flows

$$a^2\nabla\cdot\boldsymbol{u} - \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{u} = 0$$

Outline

2D potential flow: method of characteristics

- Non linear potential eq.

$$a^2 \nabla \cdot \boldsymbol{u} - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{u} = 0$$

- Irrotational flow

$$\omega = \nabla \times \boldsymbol{u} = 0 \quad \Rightarrow \quad \partial_{x_2} u_1 - \partial_{x_1} u_2 = 0$$

- System of partial differential eqs.

$$\begin{pmatrix} a^2 - u_1^2 & -u_1 u_2 \\ 0 & 1 \end{pmatrix} \partial_{x_1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -u_1 u_2 & a^2 - u_2^2 \\ -1 & 0 \end{pmatrix} \partial_{x_2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$\Rightarrow$

$$\partial_{x_1} Q + A\, \partial_{x_2} Q = 0$$

with

$$Q = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad A = \frac{1}{a^2 - u_1^2} \begin{pmatrix} -2u_1 u_2 & a^2 - u_2^2 \\ -(a^2 - u_1^2) & 0 \end{pmatrix}$$

Eigen vectors and eigen values

- Left eigen vector l and eigen value λ: $\quad l\ A = \lambda l$
- Characteristic equation: $l\ (A - \lambda\mathbb{I}) = 0 \quad \Rightarrow \quad |A - \lambda\mathbb{I}| = 0$

$$(a^2 - u_1^2)\ \lambda^2 + 2u_1 u_2\ \lambda + a^2 - u_2^2 = 0$$

- Discriminant: $\Delta = a^2(u_1^2 + u_2^2 - a^2)$
  - $M > 1 \rightarrow \Delta > 0$: hyperbolic eq.
  - $M = 1 \rightarrow \Delta = 0$: parabolic eq.
  - $M < 1 \rightarrow \Delta < 0$: elliptic eq.

- For $M > 1$ $(\varepsilon = \pm)$
  - Real eigen values: $\lambda_\varepsilon = \frac{u_1 u_2/a^2 + \varepsilon\sqrt{M^2-1}}{u_1^2/a^2 - 1}$
  - Eigen vectors: $(l_{11}\ \ l_{12})\ A = \lambda_+(l_{11}\ \ l_{12})$ and $(l_{21}\ \ l_{22})\ A = \lambda_-(l_{21}\ \ l_{22})$
    $l_{12}/l_{11} = (\lambda_+ - a_{11})/a_{21}$ and $l_{22}/l_{21} = (\lambda_- a_{11})/a_{21}$
  - After some algebra, one gets

$$l_{12}/l_{11} = \lambda_- \text{ and } l_{22}/l_{21} = \lambda_+$$

Physical interpretation of the characteristic lines

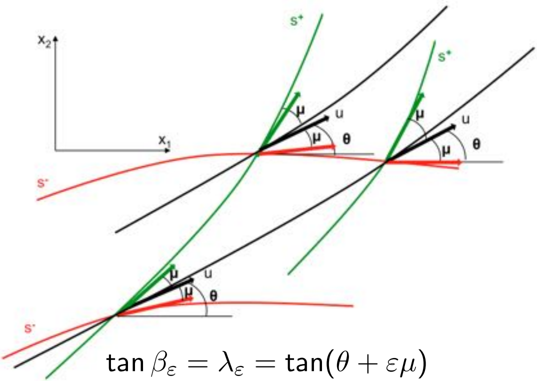
- Change of variables $\begin{cases} u_1/a = M\cos\theta \\ u_2/a = M\sin\theta \end{cases}$

$$
\begin{aligned}
\lambda_\varepsilon &= \frac{M^2\sin\theta\cos\theta + \varepsilon\sqrt{M^2-1}}{M^2\cos^2\theta - 1} \\
&= \frac{M^2\tan\theta + \varepsilon\sqrt{M^2-1}(1+\tan^2\theta)}{(\varepsilon\sqrt{M^2-1})^2 - \tan^2\theta} \; : \; 1/\cos^2\theta = 1+\tan^2\theta \\
&= \frac{(\varepsilon\sqrt{M^2-1}\tan\theta + 1)(\varepsilon\sqrt{M^2-1}+\tan\theta)}{(\varepsilon\sqrt{M^2-1}-\tan\theta)(\varepsilon\sqrt{M^2-1}+\tan\theta)} \\
&= \frac{\tan\theta + \varepsilon/\sqrt{M^2-1}}{1-\varepsilon\tan\theta/\sqrt{M^2-1}} \\
&= \frac{\tan\theta + \tan(\varepsilon\mu)}{1-\tan\theta\tan(\varepsilon\mu)} \; : \; \sin\mu = 1/M, \;\; \tan\mu = 1/\sqrt{M^2-1}
\end{aligned}
$$

$$\Rightarrow \lambda_\varepsilon = \tan(\theta + \varepsilon\mu)$$

with θ positive in the counterclockwise direction

- Limits: $\lim_{M\to\infty}\mu = 0$ and $\lim_{M\to 1^+}\mu = \pi/2$



$$\tan \beta_\varepsilon = \lambda_\varepsilon = \tan(\theta + \varepsilon\mu)$$

- The streamline at one point makes an angle θ with the x_1 axis
- Two characteristics are passing at this point: one at the angle μ above the streamline, and the other at the angle μ below the streamline
- The characteristic lines are **Mach lines**
- No discontinuity in the velocity or any other fluid property along the characteristics, Mach lines are patching lines for **continuous flows**

Characteristic eqs. for $M > 1$

- Using $LA = \Lambda L$ with $\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ and $L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$
  - The system of partial differential eqs. reads

$$L\partial_{x_1} Q + LA\, \partial_{x_2} Q = 0 \Rightarrow L\partial_{x_1} Q + \Lambda L\, \partial_{x_2} Q = 0$$

  - After explicit calculation

$$\begin{cases} l_{11}(\partial_{x_1} + \lambda_+ \partial_{x_2})u_1 + l_{12}(\partial_{x_1} + \lambda_+ \partial_{x_2})u_2 & = & 0 \\ l_{21}(\partial_{x_1} + \lambda_- \partial_{x_2})u_1 + l_{22}(\partial_{x_1} + \lambda_- \partial_{x_2})u_2 & = & 0 \end{cases}$$

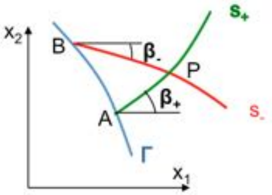
  - The terms between ( ) can be interpreted as directional derivatives in the x_1, x_2 plane. Using the characteristic directions $s_\varepsilon = (\cos\beta_\varepsilon, \sin\beta_\varepsilon)$

$$\begin{cases} l_{11} \frac{\mathrm{d}}{\mathrm{d}s_+} u_1 + l_{12} \frac{\mathrm{d}}{\mathrm{d}s_+} u_2 & = & 0 \\ l_{21} \frac{\mathrm{d}}{\mathrm{d}s_-} u_1 + l_{22} \frac{\mathrm{d}}{\mathrm{d}s_-} u_2 & = & 0 \end{cases} \quad \Rightarrow \quad \begin{cases} l_{11} + l_{12} \frac{\mathrm{d}u_2}{\mathrm{d}u_1} & = & 0 & \text{on } s_+ \\ l_{21} + l_{22} \frac{\mathrm{d}u_2}{\mathrm{d}u_1} & = & 0 & \text{on } s_- \end{cases}$$

one obtains a system of ordinary differential eqs. in the u_1, u_2 plane (hodograph plane) along the directions s_+ and s_-

Solution method

- Cauchy problem: u_1 and u_2 are given on an arc Γ and one wishes to compute the flow downstream of that arc

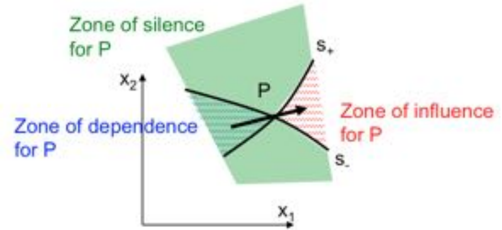


- Discretized solution
  - The characteristic slope direction λ_+ and λ_- can be computed at each point of arc Γ, in particular $\lambda_+|_A = \tan\beta_+|_A$ and $\lambda_-|_B = \tan\beta_-|_B$
  - The position of point P is defined by the intersection of the characteristic lines approximately numerically by their tangent at A and B
  - The velocity components at P can be computed by integrating (numerically) the characteristic equations

$$\begin{cases} \dfrac{u_1|_P - u_1|_A}{\Delta s_+} + \lambda_-|_A \dfrac{u_2|_P - u_2|_A}{\Delta s_+} = 0 \quad \text{on } s_+ \\ \dfrac{u_1|_P - u_1|_B}{\Delta s_-} + \lambda_+|_B \dfrac{u_2|_P - u_2|_B}{\Delta s_-} = 0 \quad \text{on } s_- \end{cases}$$

Zone of influence and zone of silence

- Consider left- and right-running characteristics through point P



  - Zone of influence for P: area between the two downstream characteristics. This region is influenced by any disturbances or information at point P
  - Zone of silence for P: all points outside the zone of influence will not be affected by disturbances or information at P
  - Zone of dependence for P: area between the two upstream characteristics. Properties at point P depend on any disturbances or information in the flow within this upstream region
- In steady supersonic flow, disturbances do not propagate upstream

Riemann invariants

- The characteristic eqs. in the hodograph plane

$$\begin{cases} 1+\lambda_- \frac{\mathrm{d}u_2}{\mathrm{d}u_1} & = & 0 & \text{on } s_+ \\ 1+\lambda_+ \frac{\mathrm{d}u_2}{\mathrm{d}u_1} & = & 0 & \text{on } s_- \end{cases}$$

  have only one solution through the point of the plane
- These solutions are called Riemann invariants

$$\begin{cases} F_1(u_1,u_2) & = & \mathit{constant} & \text{on } s_+ \\ F_2(u_1,u_2) & = & \mathit{constant} & \text{on } s_- \end{cases}$$

- After change of variables $(u_1, u_2) \to (M, \theta)$, one obtains

$$\begin{cases} \nu(M)-\theta & = & \mathit{constant} & \text{on } s_+ \\ \nu(M)+\theta & = & \mathit{constant} & \text{on } s_- \end{cases}$$

  with the the Prandtl-Meyer function $\nu(M)$ introduced in lecture 2

Proof

- Change of variables $\begin{cases} u_1 = u\cos\theta \Rightarrow du_1 = \cos\theta du - u\sin\theta d\theta \\ u_2 = u\sin\theta \Rightarrow du_2 = \sin\theta du + u\cos\theta d\theta \end{cases}$

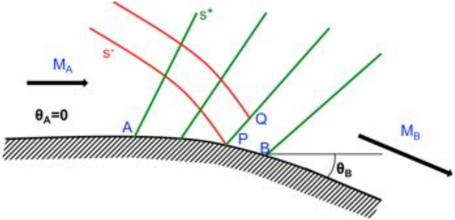
$$\begin{aligned} du_1 + \lambda_{-\varepsilon} du_2 &= 0 \\ (\cos\theta + \lambda_{-\varepsilon}\sin\theta)du + u(-\sin\theta + \lambda_{-\varepsilon}\cos\theta)d\theta &= 0 \\ (\cos(\theta-\varepsilon\mu)\cos\theta + \sin(\theta-\varepsilon\mu)\sin\theta)du & \\ +u(-\cos(\theta-\varepsilon\mu)\sin\theta + \sin(\theta-\varepsilon\mu)\cos\theta)d\theta &= 0 \\ \cot\mu\frac{1}{u}du - \varepsilon d\theta &= 0 \end{aligned}$$

- Change of variable: $\begin{cases} H = c_pT + \frac{1}{2}u^2 = (\frac{1}{(\gamma-1)M^2} + \frac{1}{2})u^2 \\ dH = (\frac{1}{(\gamma-1)M^2} + \frac{1}{2})2udu - \frac{2}{\gamma-1}u^2\frac{dM}{M^3} \\ \quad = 0 \end{cases}$

$$\begin{aligned} g(M)dM - \varepsilon d\theta &= 0: \quad \text{with } g(M) = \frac{\sqrt{M^2-1}}{M}\frac{1}{1+\frac{\gamma-1}{2}M^2} \\ \nu(M) - \varepsilon\theta &= constant: \quad \text{with } \nu(M) = \int_0^M g(M')\, dM' \end{aligned}$$

Prandtl-Meyer expansion

- Let us consider an incoming flow following a curved (convex) surface

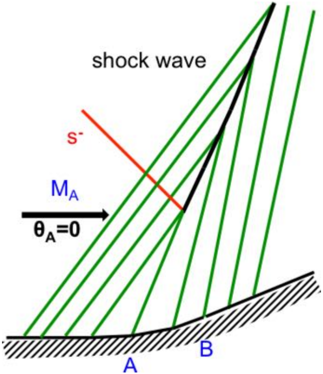


- It is possible to compute exactly the flow quantities at each point P of the wall (the angle θ is negative in the clockwise direction)
  - On s_- emanating from the uniform incoming flow region
    $$\nu(M_P) + \theta_P = \nu(M_A)$$
  - For any point Q on s_+ emanating from point P
    $$\begin{cases} \nu(M_Q) - \theta_Q & = \quad \nu(M_P) - \theta_P \\ \nu(M_Q) + \theta_Q & = \quad \nu(M_A) \quad = \quad \nu(M_P) + \theta_P \end{cases}$$

  $\rightarrow$ $\nu(M_Q) = \nu(M_P)$ and $\theta_Q = \theta_P$: the slope of s_+ is identical at all point (straight line)
- The angle $\theta + \mu$ decreases: the characteristics diverge and form a fan

Formation of a shock wave

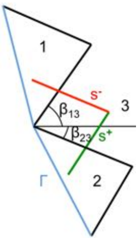
- Let us consider an incoming flow following a curved (concave) surface, the characteristics converge
  - The characteristics from the same family might therefore intersect
  - Such an intersection indicates the breakdown of the method of characteristics, because there would be too much information to calculate the flow quantities at the point of intersection



- In reality, the intersection of characteristics in not possible, it rather indicates the appearance of a shock wave
- As opposed to characteristics, shocks are patching lines for **discontinuous flows**

In practice, finite-volume representation

- In the finite-volume representation of the method of characteristics, the solution, supposed piecewise constant over elementary cells, is obtained as follows:
  - Divide the Cauchy arc Γ into a number of segments on which the flow conditions are assumed to be constant
  - Approximate by straight lines the characteristics emanating from the segment edges
  - Based on the characteristics, define a set of elementary cells on which the flow quantities will be determined
  - Compute the slopes of the characteristics



- On s^-: $\nu(M_3) + \theta_3 = \nu(M_1) + \theta_1$
- On s^+: $\nu(M_3) - \theta_3 = \nu(M_2) - \theta_2$

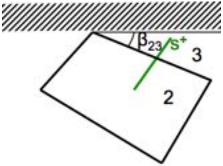
$\Rightarrow$

$$\begin{cases} \nu(M_3) & = & \frac{\nu(M_1)+\nu(M_2)}{2} + \frac{\theta_1-\theta_2}{2} \\ \theta_3 & = & \frac{\nu(M_1)-\nu(M_2)}{2} + \frac{\theta_1+\theta_2}{2} \end{cases}$$

$\Rightarrow$

$$\begin{cases} \beta_{13} & = & \frac{\theta_1+\theta_3}{2} + \frac{\mu_1+\mu_3}{2} \\ \beta_{23} & = & \frac{\theta_2+\theta_3}{2} - \frac{\mu_2+\mu_3}{2} \end{cases}$$

- When characteristics hit physical boundaries
  - Wall: flow angle imposed
  - Free jet: pressure and thus Mach number imposed (provided that isentropic assumption is valid)



Example: wall

- On s^+: $\nu(M_3) - \theta_3 = \nu(M_2) - \theta_2$
- $\theta_3 = \theta_{wall}$

$\Rightarrow$

$$\nu(M_3) = \nu(M_2) + \theta_{wall} - \theta_2$$

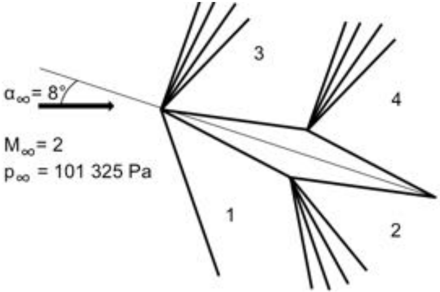
$\Rightarrow$

$$\beta_{23} = \frac{\theta_2+\theta_3}{2} - \frac{\mu_2+\mu_3}{2}$$

Outline

Exercise: forces on 2D airfoils

Consider a symmetrical diamond-shaped airfoil flying at an angle of attack to the free stream of 8° when the upstream Mach number is 2. The atmospheric pressure of the free stream is equal to 101,325 Pa. The ratio of the thickness t to the chord c of the airfoil is equal to 0.1.



Calculate the lift and drag forces exerted on the airfoil assuming that the chord is equal to 1m ($\mathrm{R} = 287$ J/ (kg K), $\gamma = 1.4$, $T_\infty = 188$K).

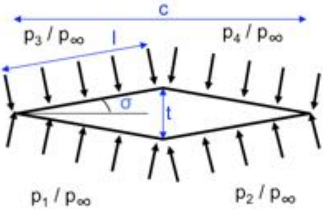
Solution

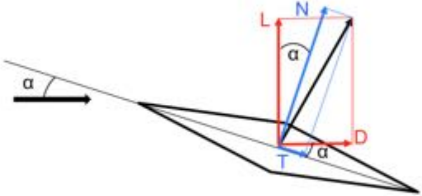
- The airfoil half-angle σ is determined from $\sigma = \arctan(t/c) = 5.7106^\circ$
- The pressure coefficient for region $i = 1, 2, 3, 4$ is defined as

$$C_{pi} = \frac{p_i - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_i}{p_\infty} - 1\right)$$

- The forces exerted on the airfoil in the local reference frame are

$$\begin{aligned} T &= (p_1 + p_3 - p_2 - p_4) l \sin\sigma = \frac{\gamma M_\infty^2 p_\infty}{2} \frac{t}{2}(C_{p1} + C_{p3} - C_{p2} - C_{p4}) \\ N &= (p_1 + p_2 - p_3 - p_4) l \cos\sigma = \frac{\gamma M_\infty^2 p_\infty}{2} \frac{c}{2}(C_{p1} + C_{p2} - C_{p3} - C_{p4}) \end{aligned}$$





- The lift and drag forces exerted on the airfoil in the flow reference frame are

$$L = -T\sin\alpha + N\cos\alpha$$
$$D = T\cos\alpha + N\sin\alpha$$

- **Region 1: oblique shock theory**
  - Deflection angle $\theta = 8^\circ + \sigma = 13.7106^\circ$
  - Shock-wave angle: $\beta = 43.6598^\circ$
  - Normal Mach: $M_{n\infty} = 1.3807$
  - Static pressure: $p_1/p_\infty = 2.0575$
  - Pressure coefficient: $C_{p1} = 0.377$
  - Static temperature: $\frac{T_1}{T_\infty} = \left[1 + \frac{2\gamma}{\gamma+1}(M_{n\infty}^2 - 1)\right]\left[\frac{2+(\gamma-1)M_{n\infty}^2}{(\gamma+1)M_{n\infty}^2}\right] = 1.2422$
  - Normal Mach number: $M_{n1}^2 = \frac{1+\frac{\gamma-1}{2}M_{n\infty}^2}{\gamma M_{n\infty}^2 - \frac{\gamma-1}{2}} = 0.7479$
  - Tangentiel Mach number: $M_{t1} = M_{t\infty}\sqrt{\frac{T_\infty}{T_1}} = 1.2982$
  - Mach number: $M_1 = \sqrt{M_{n1}^2 + M_{t1}^2} = 1.4982$

- **Region 2: Prandtl-Meyer expansion**
  - Mach number: $\nu(M_2) = \nu(M_1) + 2\sigma = 23.2734^\circ \Rightarrow M_2 = 1.8889$
  - Static pressure: $\frac{p_2}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow \frac{p_2}{p_\infty} = \frac{p_2}{p_1}\frac{p_1}{p_\infty} = 1.1436$
  - Pressure coefficient: $C_{p2} = 0.051$

- **Region 3: Prandtl-Meyer expansion**
  - Mach number: $\nu(M_3) = \nu(M_\infty) + \alpha - \sigma = 28.6684^\circ \Rightarrow M_3 = 2.0844$
  - Static pressure: $\frac{p_3}{p_\infty} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_3^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.8766$
  - Pressure coefficient: $C_{p3} = -0.0441$
- **Region 4: Prandtl-Meyer expansion**
  - Mach number: $\nu(M_4) = \nu(M_3) + 2\sigma = 40.0896^\circ \Rightarrow M_4 = 2.5417$
  - Static pressure: $\frac{p_4}{p_\infty} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_4^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.4292$
  - Pressure coefficient: $C_{p4} = -0.2039$
- **Forces**

$$\begin{aligned} C_T &= \tfrac{1}{2}\frac{t}{c}(C_{p1} + C_{p3} - C_{p2} - C_{p4}) = 0.05(0.377 - 0.0441 - 0.051 + 0.2039) = 0.02429 \\ C_N &= \frac{1}{2}(C_{p1} + C_{p2} - C_{p3} - C_{p4}) = 0.5(0.377 + 0.051 + 0.0441 + 0.2039) = 0.338 \end{aligned}$$

$$\begin{aligned} T &= \frac{\gamma M_\infty^2 p_\infty}{2} C_T = 6891 \ \mathrm{N/m^2} \\ N &= \frac{\gamma M_\infty^2 p_\infty}{2} C_N = 95\ 893 \ \mathrm{N/m^2} \end{aligned}$$

$$\begin{aligned} L &= -T\sin\alpha + N\cos\alpha = 94\ 001\ \mathrm{N/m^2} \\ D &= T\cos\alpha + N\sin\alpha = 20\ 170\ \mathrm{N/m^2} \end{aligned}$$

For a supersonic inviscid flow over an infinite wing, the drag per unit span is finite. This wave drag is inherently related to the loss of total pressure and increase of entropy across the oblique shock waves created by the airfoil.