Flight Dynamics and Control

Lecture 1: Introduction

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Reference material

- Lecture Notes
- Flight Dynamics Principles, M.V. Cook, Arnold, 1997
What is it about?
Introduction

• The study of the mechanics and dynamics of flight is the means by which:
  – We can design an airplane to accomplish efficiently a specific task
  – We can make the task of the pilot easier by ensuring good handling qualities
  – We can avoid unwanted or unexpected phenomena that can be encountered in flight
The pilot has direct control only of the Flight Control System. However, he can tailor his inputs to the FCS by observing the airplane’s response while always keeping an eye on the task at hand.
Control Surfaces

- Aircraft control is accomplished through control surfaces and power
  - Ailerons
  - Elevators
  - Rudder
  - Throttle
- Control deflections were first developed by the Wright brothers from watching birds
The Flyer did not have separate control surfaces. The trailing edges of the windtips could be bent by a system of cables.
Modern control surfaces

- Rudder
- Elevator
- Canard
- Aileron Up
- Aileron Down
Other devices

- Flaps
- Spoilers
- Airbreak
- Slats
- Vectored thrust
Combinations

- Flaperon
- Elevon
- Split rudder/airbrake
- Spoileron
- Deceleron
- Rudderon
Mathematical Model

Input
- Aileron
- Elevator
- Rudder
- Throttle

Flight Condition

Aircraft equations of motion

Output
- Displacement
- Velocity
- Acceleration

Atmospheric Condition
Aircraft degrees of freedom

Six degrees of freedom:

3 displacements
- $x$: horizontal motion
- $y$: side motion
- $z$: vertical motion

3 rotations
- Around $x$: roll
- Around $y$: pitch
- Around $z$: yaw

$\mathbf{U}$: resultant linear velocity, cg: centre of gravity
$\mathbf{w}$: resultant angular velocity
Aircraft frames of reference

• There are many possible coordinate systems:
  – Inertial (immobile and far away)
  – Earth-fixed (rotates with the earth’s surface)
  – Vehicle carried vertical frame (fixed on aircraft cg, vertical axis parallel to gravity)
  – Air-trajectory (fixed on aircraft cg, parallel to the direction of motion of the aircraft)
  – Body-fixed (fixed on aircraft cg, parallel to a geometric datum line on the aircraft)
  – Stability axes (fixed on aircraft cg, parallel to a reference flight condition)
  – Others
Airplane geometry

\[ s = b / 2 \]

\[ c(y) \]

\[ x(y) \]
Airplane references (1)

- **Standard mean chord (SMC)**
  \[ \bar{c} = \frac{\int_{-s}^{s} c(y)\,dy}{\int_{-s}^{s} dy} \]

- **Mean aerodynamic chord (MAC)**
  \[ \bar{c} = \frac{\int_{-s}^{s} c^2(y)\,dy}{\int_{-s}^{s} c(y)\,dy} \quad x_{MAC} = \frac{\int_{-s}^{s} c(y)x(y)\,dy}{\int_{-s}^{s} c(y)\,dy} \]

- **Wing area**
  \[ S = b\bar{c} \]

- **Aspect Ratio**
  \[ AR = \frac{b^2}{S} \]
Airplane references (2)

- Centre of gravity \( (cg) \)
- Tailplane area \( (S_T) \)
- Tail moment arm \( (l_T) \)
- Tail volume ratio: A measure of the aerodynamic effectiveness of the tailplane

\[
\bar{V}_T = \frac{S_T l_T}{\bar{S} \bar{c}}
\]
Airplane references (3)

- Fin moment arm \( (l_F) \)
- Fin volume ratio

\[
\bar{V}_F = \frac{S_F l_F}{S \bar{c}}
\]
Aerodynamic Reference Centres

• Centre of pressure (\(cp\)): The point at which the resultant aerodynamic force \(F\) acts. There is no aerodynamic moment around the \(cp\).

• Half-chord: The point at which the aerodynamic force due to camber, \(F_c\), acts.

• Quarter-chord (or aerodynamic centre): The point at which the aerodynamic force due to angle of attack, \(F_a\), acts. The aerodynamic moment around the quarter-chord, \(M_0\), is constant with angle of attack.
Airfoil with centres

By placing all of the lift and drag on the aerodynamic centre we move the lift and drag due to camber from the half-chord to the quarter chord. This is balanced by the moment $M_0$. 
Full description of aircraft motion

• The static stability analysis presented in the aircraft design lectures is good for the preliminary design of aircraft

• Aircraft flight is a dynamic phenomenon:
  – Every control input or external excitation results in a dynamic response
  – The dynamic response may be oscillatory and have a single or several frequency components
  – The dynamic response may be damped (stable) or undamped (unstable)

• The modelling of this dynamic response necessitates the derivation of the full equations of motion of the aircraft
Nomenclature

- Here is a definition of the degrees of freedom of an aircraft and the forces and moments acting on it.
- All degrees of freedom are relative to the aircraft’s centre of gravity and use aircraft geometrical axes.

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$, $U$, $X$</td>
<td>translation, velocity and force applied in the direction parallel to the axis of the fuselage</td>
</tr>
<tr>
<td>$y$, $V$, $Y$</td>
<td>translation, velocity and force applied in the direction perpendicular to the plane of symmetry of the aircraft</td>
</tr>
<tr>
<td>$z$, $W$, $Z$</td>
<td>translation, velocity and force applied in the direction perpendicular to both $x$ and $y$</td>
</tr>
<tr>
<td>$p$, $L$</td>
<td>angular velocity and moment in roll direction</td>
</tr>
<tr>
<td>$q$, $M$</td>
<td>angular velocity and moment in pitch direction</td>
</tr>
<tr>
<td>$r$, $N$</td>
<td>angular velocity and moment in yaw direction</td>
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</tbody>
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Body and axes

Axis system

Could be any body but in this case it is an aircraft of mass $m$.

For the moment it is a flexible body

Any point $p$ on the body can have a velocity and acceleration with respect to the c.g.
Vector notation

• We define the following vector notation

\[
x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad w = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad U = \begin{bmatrix} U \\ V \\ W \end{bmatrix}, \quad F = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad M = \begin{bmatrix} L \\ M \\ N \end{bmatrix}
\]

\[
u = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}
\]

• Noting that \( u \) and \( a \) are velocities and accelerations with respect to the center of gravity
Developing the equations of motion

• All equations of motion of dynamic systems can be derived using Newton’s Second Law.
• Two sets of equations are derived:
  – Sum of forces acting on the system (internal and external) are equal to its mass times its acceleration
  – Sum of moments acting on the system (internal and external) are equal to its moment of inertia times its angular acceleration
• Therefore, the object of the derivation is to estimate the accelerations (linear and angular of the aircraft)
• As usual, the same equations of motion can be obtained using Lagrange’s equation (i.e. conservation of energy)
Local velocities (1)

- The local velocity vector $\mathbf{u}$ is given simply by
  \[ \mathbf{u} = \dot{\mathbf{x}} + \mathbf{w} \times \mathbf{x} \]

- Substituting for the vector definitions
  \[
  \mathbf{u} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}
  \]

- Where $\times$ denotes the vector (cross) product and
  \[
  \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} i & j & k \\ p & q & r \\ x & y & z \end{vmatrix}
  \]
Local velocities (2)

• The equations for the local velocities at point $p(x,y,z)$ are

\[ u = \dot{x} - ry + qz \]
\[ v = \dot{y} - pz + rx \]
\[ w = \dot{z} - qx + py \]

• Now assume that the body is rigid, i.e. no parts of it are moving with respect to the c.g
Total local velocities

• This gives \( \dot{x} = \dot{y} = \dot{z} = 0 \) therefore
  \[
u = -ry + qz
  \]
  \[
  u = w \times x, \text{ or, } v = -pz + rx
  \]
  \[
w = -qx + py
  \]

• The total local velocities \( u' = u + U \) at \( p(x, y, z) \) are given by
  \[
u' = U + u = U - ry + qz
  \]
  \[
v' = V + v = V - pz + rx
  \]
  \[
w' = W + w = W - qx + py
  \]
Similarly, the local accelerations at point $p(x,y,z)$ are given by

$$a = \ddot{u} + w \times u$$  \hspace{1cm} (2)

Substituting for the vector definitions

$$a = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -\dot{y} + \dot{q}z \\ -\dot{p}z + \dot{r}x \\ -\dot{q}x + \dot{p}y \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{vmatrix} i & j & k \\ p & q & r \\ -\dot{r}y + qz & -\dot{p}z + \dot{r}x & -\dot{q}x + \dot{p}y \end{vmatrix}$$
Local accelerations (2)

- Carrying out all the algebra leads to

\[ a_x = -x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q}) \]
\[ a_y = x(pq + \dot{r}) - y(p^2 + r^2) + z(qr - \dot{p}) \]
\[ a_z = x(pr - \dot{q}) + y(qr + \dot{p}) - z(p^2 + q^2) \]

- Remembering that this is only part of the acceleration of point \( p \). The acceleration of the centre of gravity must be added.
Total local acceleration

• The total local acceleration at point \( p(x,y,z) \) is defined as

\[
a' = \hat{U} + w \times U + a
\]

• So that, finally

\[
\begin{align*}
a'_x &= \dot{U} - rV + qW - x\left(q^2 + r^2\right) + y\left(pq - \dot{r}\right) + z\left(pr + \dot{q}\right) \\
a'_y &= \dot{V} - pW + rU + x\left(pq + \dot{r}\right) - y\left(p^2 + r^2\right) + z\left(qr - \dot{p}\right) \\
a'_z &= \dot{W} - qU + pV + x\left(pr - \dot{q}\right) + y\left(qr + \dot{p}\right) - z\left(p^2 + q^2\right)
\end{align*}
\]
Example

- A pilot in an aerobatic aircraft performs a loop in 20s at a steady velocity of 100m/s. His seat is located 5m ahead of, and 1m above, the c.g. What total normal load factor does he experience at the top and the bottom of the loop?
Solution

Movement only in the plane of symmetry:

\[ V = \dot{p} = p = r = 0 \]

Normal acceleration:

\[ a'_z = \dot{W} - qU + x\dot{q} - zq^2 \]

For a steady manoeuvre:

\[ \dot{W} = \dot{q} = 0 \]

Pitch rate:

\[ q = \frac{2\pi}{20} = 0.314 \text{rad/s} \]
Solution (2)

• Substituting into equation for normal acceleration at the seat:

\[ a'_z = -qU - zq^2 = -0.314 \times 100 - (-1) \times 0.314^2 = -31.3 \text{m/s}^2 \]

• Normal load factor definition:

\[ n' = \frac{a'_z}{g} = \frac{31.3}{9.81} = 3.19 \]

• Total normal load factor at top of loop:

\[ n = n' - 1 = 2.19 \]

• Total normal load factor at bottom of loop:

\[ n = n' + 1 = 4.19 \]
Generalized Force Equations

• Assume that point \( p(x, y, z) \) has a small mass \( dm \).

• Applying Newton’s 2\(^{nd}\) law to the entire body yields

\[
\int_{\text{Vol}} a' \, dm = F \quad (5)
\]

• where the subscript \( \text{Vol} \) denotes that the integral is taken over the entire volume.
Force equations (2)

• Remember from equation (3) that
  \[ a' = \dot{U} + w \times U + a \]

• Substituting from equations (2) and (1)
  \[ a' = \dot{U} + w \times U + \dot{w} \times x + w \times (w \times x) \]  \(\text{(6)}\)

• Putting this last result back into Newton’s 2\textsuperscript{nd} Law, equation (5)
  \[ \int_{\text{Vol}} (\dot{U} + w \times U + \dot{w} \times x + w \times (w \times x)) \, dm = F \]
Centre of gravity

• As far as the integral over the volume is concerned, $w$ and $U$ are constants.

• The generalized force equation becomes

$$
\dot{U} \int dm + w \times U \int dm + \dot{w} \times \int x dm + w \times \left( w \times \int x dm \right) = F
$$

• The definition of the centre of gravity is

$$
\int x dm = 0
$$

• The force equation becomes

$$
m(\dot{U} + w \times U) = F \quad (7)
$$
Generalized Moment Equations

• The angular acceleration of point \( p(x,y,z) \) around the centre of gravity is given by
  \[ x \times a' \]

• Again, use Newton’s second law, this time in moment form, to obtain
  \[ \int_{\text{Vol}} x \times a' \, dm = M \]  
  \[ \text{(8)} \]

• Substitute from equation (6)
  \[ \int_{\text{Vol}} x \times (\dot{U} + w \times U + \dot{w} \times x + w \times (w \times x)) \, dm = M \]
Center of gravity

- Using the definition of the centre of gravity, the moment equation becomes

\[
\int_{\text{Vol}} x \times (\dot{w} \times x) \, dm + \int_{\text{Vol}} x \times [w \times (w \times x)] \, dm = M
\]

- Now remember the matrix form of the cross product

\[
x \times w = Xw, \quad w \times x = X^T w
\]

where \( X = \begin{bmatrix}
0 & -z & y \\
z & 0 & -x \\
-y & x & 0
\end{bmatrix} \)
Moments of inertia

- The first term in the moment equation becomes

\[
\int_{\text{Vol}} x \times (\dot{w} \times x) \, dm = \int_{\text{Vol}} X X^T \dot{w} \, dm = \left( \int_{\text{Vol}} X X^T \, dm \right) \dot{w}
\]

- where

\[
I_c = \int_{\text{Vol}} X X^T \, dm = \int_{\text{Vol}} \begin{bmatrix}
y^2 + z^2 & -xy & -xz \\
-xy & x^2 + z^2 & -yz \\
-xz & -yz & x^2 + y^2
\end{bmatrix} \, dm
\]

- is the system’s’s inertia matrix
Moments of inertia (2)

• The individual moments and products of inertia are defined as

\[
I_x = \int \left( y^2 + z^2 \right) dm, \quad I_y = \int \left( x^2 + z^2 \right) dm, \quad I_z = \int \left( x^2 + y^2 \right) dm
\]

\[
I_{xy} = \int xy \, dm, \quad I_{xz} = \int xz \, dm, \quad I_{yz} = \int yz \, dm
\]

• So that the inertia matrix becomes

\[
\mathbf{I}_c = \begin{bmatrix}
I_x & -I_{xy} & -I_{xz} \\
-I_{xy} & I_y & -I_{yz} \\
-I_{xz} & -I_{yz} & I_z
\end{bmatrix}
\] (8)
Moment equation

- Using the definition of the inertia matrix, the first term in the moment equation becomes simply
  \[ \int_{\text{Vol}} \mathbf{x} \times (\dot{\mathbf{w}} \times \mathbf{x}) \, dm = I_c \dot{\mathbf{w}} \]

- Similarly, the second term is
  \[ \int_{\text{Vol}} \mathbf{x} \times [\mathbf{w} \times (\mathbf{w} \times \mathbf{x})] \, dm = \mathbf{w} \times (I_c \mathbf{w}) \]

- The full moment equation becomes
  \[ I_c \dot{\mathbf{w}} + \mathbf{w} \times (I_c \mathbf{w}) = \mathbf{M} \quad (9) \]
Complete equations of motion

• Assembling equations (7) and (9) we get the complete equations of motion

\[ m(\dot{U} + w \times U) = F \]
\[ I_c \dot{w} + w \times (I_c w) = M \]  \hspace{1cm} (10)

• This is a set of 6 equations of motion with 6 unknowns, \( U, V, W, p, q, r \).

• They are nonlinear Ordinary Differential Equations.
Scalar form

• Substituting for the definitions of $I_c$, $U$, $w$, $F$ and $M$ we get a nicer form

\[
m\left(\dot{U} - rV + qW\right) = X
\]
\[
m\left(\dot{V} - pW + rU\right) = Y
\]
\[
m\left(\dot{W} - qU + pV\right) = Z
\]

\[
I_x \dot{p} - (I_y - I_z)qr + I_{xy} (pr - \dot{q}) - I_{xz} (pq + \dot{r}) + I_{yz} (r^2 - q^2) = L
\]
\[
I_y \dot{q} + (I_x - I_z)pr + I_{yz} (pq - \dot{r}) + I_{xz} (p^2 - r^2) - I_{xy} (qr + \dot{p}) = M
\]
\[
I_z \dot{r} - (I_x - I_y)pq - I_{yz} (pr + \dot{q}) + I_{xz} (qr - \dot{p}) + I_{xy} (q^2 + p^2) = N
\]
Symmetric aircraft

- Consider an aircraft that is symmetric about the $x$-$z$ plane.
- For every point $p(x, y, z)$ with mass $dm$, there is a point $p(x, -y, z)$ with mass $dm$.
- It follows that
  \[ I_{xy} = \int_{\text{Vol}} xy \, dm = 0 \]
- Similarly,
  \[ I_{yz} = \int_{\text{Vol}} yz \, dm = 0 \]
The elementary mass moment $xydm$ around the CG is cancelled by the elementary mass moment $x(-y)dm$. 
Asymmetric Aircraft

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Inside The Boomerang

Ruttan Bumerang

Viewed from above or below, the Boomerang really does look like its namesake. Its forward-swept 101.7-sq.-ft. wing spans 36.7 ft. and has a 13.2 aspect ratio. Inside, the 1000-pound payload cabin has five seats that are offset for added shoulder room, and the rear-most seat becomes a bed.

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Symmetric aircraft (2)

- For symmetric aircraft, the equations of motion become

\[
\begin{align*}
  m \left( \dot{U} - rV + qW \right) &= X \\
  m \left( \dot{V} - pW + rU \right) &= Y \\
  m \left( \dot{W} - qU + pV \right) &= Z \\
  I_x \dot{p} - (I_y - I_z) qr - I_{xz} \left( pq + \dot{r} \right) &= L \\
  I_y \dot{q} + (I_x - I_z) pr + I_{xz} \left( p^2 - r^2 \right) &= M \\
  I_z \dot{r} - (I_x - I_y) pq + I_{xz} \left( qr - \dot{p} \right) &= N
\end{align*}
\]
Discussion of the equations

- If we can solve for $U, V, W, p, q, r$ as functions of time, then we know the complete time history of the motion of the aircraft.
- Unfortunately, terms such as $rU, pV, qW$, etc and $pq, r^2, qr$ etc are nonlinear.
- Furthermore, we have only defined the inertial loads up to now.
- We have not said anything about the external loads acting on the aircraft.
External Forces and Moments

• There are five sources of external forces and moments:
  – **Aerodynamic**
  – **Gravitational**
  – **Controls**
  – **Propulsion**
  – Atmospheric **Disturbances**
External Forces and moments

• The full equations of motion in the presence of external forces and moments are

\[
\begin{align*}
    m(\dot{U} - rV + qW) &= X_a + X_g + X_c + X_p + X_d \\
    m(\dot{V} - pW + rU) &= Y_a + Y_g + Y_c + Y_p + Y_d \\
    m(\dot{W} - qU + pV) &= Z_a + Z_g + Z_c + Z_p + Z_d \\
    I_x \dot{p} - (I_y - I_z)qr - I_{xz}(pq + \dot{r}) &= L_a + L_g + L_c + L_p + L_d \\
    I_y \dot{q} + (I_x - I_z)pr + I_{xz}(p^2 - r^2) &= M_a + M_g + M_c + M_p + M_d \\
    I_z \dot{r} - (I_x - I_y)pq + I_{xz}(qr - \dot{p}) &= N_a + N_g + N_c + N_p + N_d
\end{align*}
\]