Flight Dynamics and Control

Lecture 3: Solution of the Equations of Motion

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Solution of the Equations of Motion

• We have seen that the equations of motion of a rigid aircraft can be of the form:

\[
\begin{bmatrix}
\dot{v} \\
\dot{p} \\
\dot{r} \\
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
-0.0565 & 29.072 & -175.610 & 9.6783 & 1.6022 \\
-0.0601 & -0.7979 & 0.2996 & 0 & 0 \\
9.218 \times 10^{-3} & -0.0179 & -0.1339 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\p \\r \\\phi \\\psi
\end{bmatrix} +
\begin{bmatrix}
-0.2678 & 2.0092 \\
4.6982 & 0.7703 \\
0.0887 & -1.3575 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\p \\r \\\phi \\\psi
\end{bmatrix}
\]

• This is state space form, \( \dot{x} = Ax + Bu \), where \( x \) are the system states and \( u \) are the system inputs

\[
x = [v \ p \ r \ \phi \ \psi]^T, \ u = [\xi \ \zeta]^T
\]
How to solve

• The equations can be re-written in the form

\[ \dot{x} - Ax = Bu \]

• And pre-multiplied by \( e^{-At} \) yielding

\[ e^{-At} \dot{x} - e^{-At} Ax = e^{-At} Bu \]

• The left-hand side of these equations is in fact \( \frac{d}{dt}(e^{-At}x) \), so that

\[ \frac{d}{dt}(e^{-At}x) = e^{-At}Bu \]
General Solution

• In this form, the state space equations are ‘variables separable’ so that they can be easily integrated as

\[
\int_0^t d\left( e^{-At}x \right) = \int_0^t e^{-A\tau}Bu(\tau)d\tau
\]

• So that

\[
e^{-At}x(t) - e^{-A0}x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau \quad \text{or} \quad e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau
\]

• And, finally,

\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
\]
Evaluating the general solution

• The evaluation of the general solution is not trivial. There are two major difficulties:
  – The matrix exponential $e^{At}$
  – The input function integral $\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

• The matrix exponential is usually called the ‘state transition matrix’

• First we will discuss some properties of matrix exponentials
Matrix exponentials

• A matrix exponential is not equal to the exponential of all the elements of the matrix.
  - If \( E = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), then \( e^E \neq \begin{pmatrix} e^1 & e^2 \\ e^3 & e^4 \end{pmatrix} \)

• However, if \( E = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \), i.e. it is diagonal, then
  \[
eq \begin{pmatrix} e^1 & 0 \\ 0 & e^4 \end{pmatrix} \]

• In general, the matrix exponential can be calculated either using an eigensolution or a series approximation.
Matrix exponential from eigensolution

• Consider that the matrix $E$ has $n$ eigenvalues, $\lambda_i$, and $n$ eigenvectors, $v_i$.

• Then, the matrix exponential is given by

$$ e^E = \left(v_1 \quad \cdots \quad v_n\right) \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n} \end{pmatrix} \left(v_1 \quad \cdots \quad v_n\right)^{-1} $$

• Or $e^E = V e^L V^{-1}$ if $V$ is a matrix whose columns are the eigenvectors and $L$ is the diagonal matrix whose elements are the eigenvalues.
Example

• Consider the state space system

\[ \dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x \]

• Let

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

• Calculate the state transition matrix \( e^{At} \)
Solution

• The eigenvalues of $A$ are $i$ and $-i$ and the eigenvectors are
  
  $$V = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

• The eigenvalues of $At$ are $it$ and $-it$ and the eigenvectors are those of $A$.

• Then,
  
  $$e^{At} = \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right)^{-1} = \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right) \left( \begin{array}{cc} 1/2 & -i/2 \\ i/2 & 1/2 \end{array} \right)$$

• or
  
  $$e^{At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$
Series form

• In cases where the input is zero, i.e. $u(t) = 0$, the expression for the solution of the state space equations can be written in a series form as:

$$x(t) = V e^{Lt} V^{-1} x(0) = Ve^{Lt} c$$

• or

$$x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} c_i$$

• Where $c_i$ is the $i$th element of vector $c = V^{-1} x(0)$. 
Input Integral

- The form of the input integral depends on the form of the function \( u(t) \). There can be no general expression for it.

- As was done with the unforced solution, the expression for the integral can be written in series form in the following way:

\[
\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t Ve^{L(t-\tau)}V^{-1}Bu(\tau)d\tau = \int_0^t Ve^{L(t-\tau)}d(\tau)d\tau
\]

- Or

\[
\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \sum_{i=1}^n \int_0^t v_i e^{\lambda_i(t-\tau)}d_i(\tau)d\tau
\]

- Where \( d_i(\tau) \) is the \( i \)th element of vector \( d=V^{-1}Bu(\tau) \).
Therefore, the full response of the aircraft can be written in series form as

\[ x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} c_i + \sum_{i=1}^{n} \int_0^{t} v_i e^{\lambda_i (t-\tau)} d_i(\tau) d\tau \]

where \( c_i \) is the \( i \)th element of \( c = V^{-1}x(0) \) and \( d_i(\tau) \) is the \( i \)th element of \( d = V^{-1}Bu(\tau) \).

Of course \( u(t) \) must be selected in order to calculate the full response.

The usual choices are elementary input functions: step, ramp, impulse.
Step response

- If the input signal is a constant \( u(t) = u(0) \) and \( x(0) = 0 \) then the response of the system is called the ‘step response’.
- The input integral becomes
  \[
  \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = \sum_{i=1}^n \int_0^t v_i e^{\lambda_i(t-\tau)}d_i(0)\,d\tau
  \]
  \[
  = \sum_{i=1}^n v_i \left. \frac{d_i(0)}{-\lambda_i} e^{\lambda_i(t-\tau)} \right|_{\tau=0} = -\sum_{i=1}^n v_i \frac{d_i(0)}{\lambda_i} \left(1 - e^{\lambda_i t}\right)
  \]
- So that the full solution is
  \[
  x = -\sum_{i=1}^n v_i \frac{d_i(0)}{\lambda_i} \left(1 - e^{\lambda_i t}\right)
  \]
Step response of F4 longitudinal equations

The response is 0 at $t=0$. A short period (high frequency) response is damped out quickly, followed by a long period (low frequency) response that lasts 10 minutes.
Step response of F4 lateral equations

The response is 0 at $t=0$. There is no long period response, only a short period (high frequency) response that lasts under 1 minute.
Ramp Response

• If the input increases linearly from zero and \( x(0) = 0 \) the system’s response is called the ramp response.

• Write the input as \( u = mt \).

• The input integral becomes

\[
\int_0^t e^{A(t-\tau)} Bu(\tau) \, d\tau = \sum_{i=1}^n \int_0^t v_i e^{\lambda_i (t-\tau)} d_i \tau \, d\tau
\]

\[
= \sum_{i=1}^n v_i \frac{d_i}{\lambda_i^2} \left(( -\lambda_i \tau - 1) e^{\lambda_i (t-\tau)} \right) \bigg|_{\tau=0}^{\tau=t} = -\sum_{i=1}^n v_i \frac{d_i}{\lambda_i^2} \left(1 + \lambda_i t - e^{\lambda_i t} \right)
\]

• Where \( d = V^{-1} Bm \), so that

\[
x = -\sum_{i=1}^n v_i \frac{d_i}{\lambda_i^2} \left(1 + \lambda_i t - e^{\lambda_i t} \right)
\]
What exactly is a ramp?

- Step inputs can result in unwanted short period oscillations.
- A ramp input can have the same effect as a step input but this effect is achieved gradually (over $t_0$ seconds)
Important point about ramp response

• A ramp response that keeps rising makes no sense.
• A ramp response is usually followed by a step
• The force input is of the form

\[ u = \begin{cases} 
  mt & \text{if } t \leq t_0 \\
  mt_0 & \text{if } t > t_0 
\end{cases} \]

• The solution of the equations of motion must be evaluated in these intervals
Solution to ramp followed by step

- The solution is of the form

\[
x = \begin{cases} 
- \sum_{i=1}^{n} v_i \frac{d_i}{\lambda_i^2} \left( 1 + \lambda_i t - e^{\lambda_i t} \right) & \text{if } t \leq t_0 \\
\sum_{i=1}^{n} v_i e^{\lambda_i (t-t_0)} c_i - \sum_{i=1}^{n} v_i \frac{d_i t_0}{\lambda_i} \left( 1 - e^{\lambda_i (t-t_0)} \right) & \text{if } t > t_0 
\end{cases}
\]

- Where, in this case, \( c = V^{-1}x(t_0) \) and \( d = V^{-1}Bm \).
Ramp response of F4 longitudinal equations

The response is 0 at \( t=0 \). This time there is no short period oscillation, just a gentle increase up to \( t_0=20 \text{s} \). The long period oscillation is as long as before but the amplitude is slightly smaller.
Ramp response of F4 lateral equations

The response is 0 at $t=0$. The short period response is replaced by a gentle almost linear rise. When the short period response does occur, it has a very low amplitude.
Impulse Response

• An impulse response is usually defined as the response of a system to the Dirac delta function, $\delta(t)$.

• The Dirac delta is defined as:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

• And has the properties that:

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \text{ and } \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0)$$
Solution to impulsive input

- The initial conditions are still zero, i.e. \( x(0)=0 \).
- The input is of the form \( u=md(t) \).
- The input integral becomes:

\[
\int_{0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau = \sum_{i=1}^{n} \int_{0}^{t} v_i e^{\lambda_i(t-\tau)} d_i \delta(\tau) d\tau
\]

\[
= \sum_{i=1}^{n} v_i d_i e^{\lambda_i t}
\]

- Where \( d=V^{-1}Bm \) and \( m=[1 \ 0]^T \) or \( m=[0 \ 1]^T \).
- Impulse responses should only be applied to one input at a time.
Impulse response of F4 longitudinal equations

The response is 0 at $t=0$. Both short and long period oscillations are visible, the former lasting for 10 seconds the latter for 600 seconds.
The response is 0 at $t=0$. A short period vibratory response dies away after 50 seconds. A non-vibratory response flattens out (at a non-zero steady state) after 300 seconds.
Modes of vibration

- The longitudinal and lateral results demonstrate several modes of vibration:
  - Longitudinal modes:
    - Short Period Oscillation
    - Phugoid
  - Lateral modes:
    - Spiral mode
    - Roll subsidence
    - Dutch roll
Phugoid oscillations

• Notice that the application of ramp input only removed the short period oscillations and did not affect the long period ones.
• These oscillations are termed phugoids and they occur only in the longitudinal direction.
• Phugoid periods:
  – Microlight aircraft: 15-25s
  – Light aircraft: over 30s
  – Jet aircraft: minutes
• Phugoids are neutralized by re-trimming the aircraft in the new flight condition.
Phugoid Videos
Cause of Phugoids

- Phugoids are direct results of elevator deflection around a trimmed position.
- The resulting pitch change will cause the aircraft to tip either nose up or nose down.
- If it tips nose down it will gain speed, therefore lift and tip nose up again.
- If it tips nose up it will lose speed, therefore lift and tip nose down.
- The angle of attack does not change - the aircraft remains tangent to the flight path.
- The oscillation has very low damping and can last for a long time.
Phugoid mode approximation

• Lanchester model:
  – Aircraft initially in steady flight
  – Total energy of aircraft remains the same
  – The incidence is constant
  – The thrust balances the drag
  – The motion is slow so that pitch rate effects can be ignored
Energy conservation: \[ \frac{1}{2} m V_0^2 = \frac{1}{2} m V^2 + mgh = \text{const}. \]

So that: \[ V^2 = V_0^2 - 2gh \]

Lift coefficient conservation: \[ L = mg - \rho ghSC_L \]
Lanchester model (2)

- Therefore, the total force in the horizontal direction is given by:
  \[ m\ddot{h} = L\cos\theta - mg \approx L - mg \]

- Substituting from the lift equation yields:
  \[ m\ddot{h} + \left( \frac{\rho g S \mathcal{C}_L}{m} \right) h = 0 \]

- The frequency of the phugoid is then
  \[ \omega_p = \sqrt{\frac{\rho g S \mathcal{C}_L}{m}} = \frac{g\sqrt{2}}{V_0} \]
Better approximation

• A better approximation can be obtained from the longitudinal equations of motion.
• Only the equations for \( u \) and \( \theta \) are retained, along with the conditions

\[
\dot{w} = \dot{q} = 0
\]

• Then,

\[
\xi_p \omega_p = \frac{g C_D}{C_L V_0}, \quad \omega_p = \frac{g \sqrt{2}}{V_0}
\]
More about Phugoids

- Phugoid period increases with airspeed. Phugoid damping increases with airspeed.
- Compressibility effects
- Period and damping for a Boeing 747 at several altitudes and Mach numbers

\[ N_{\text{half}} = \text{number of periods until the amplitude is halved} \]
Effect of static stability margin

Small static margins decrease the period and decrease the damping of Phugoids. The choice of static margin must be balanced also with the desired degree of static stability.
Short period oscillations

- Short period oscillations are driven by the angle of attack (in French they are called oscillations d’ incidence).
- Speed changes are negligible, $u=0$.
- It is essentially a 2-DOF mechanism involving $w$ and $q$.
- They occur after abrupt input changes. Slower input changes do not cause significant short period oscillations.
Short period approximation

• The longitudinal equations of motion during a short period oscillation can be simplified to:

\[
\begin{bmatrix}
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
z_w & z_q & z_\theta \\
m_w & m_q & m_\theta \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
w \\
q \\
\theta
\end{bmatrix} +
\begin{bmatrix}
z_\eta \\
m_\eta
\end{bmatrix}\eta
\]

• Where \( z_w, m_w \) etc are constants.
• We can further simplify by assuming that changes in \( \theta \) do not affect the derivatives in \( q \) and \( w \):

\[
\begin{bmatrix}
\dot{w} \\
\dot{q}
\end{bmatrix} =
\begin{bmatrix}
z_w & z_q \\
m_w & m_q
\end{bmatrix}
\begin{bmatrix}
w \\
q
\end{bmatrix} +
\begin{bmatrix}
z_\eta \\
m_\eta
\end{bmatrix}\eta
\]
Short period approximation

- Substituting from the longitudinal equations of motion yields:

\[
\begin{bmatrix}
\dot{w} \\
\dot{q}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\tilde{Z}_w}{m - \tilde{Z}_w} & \frac{\tilde{Z}_q + mU_e}{m - \tilde{Z}_w} \\
\frac{\tilde{M}_w}{I_y} & \frac{\tilde{M}_q}{I_y}
\end{bmatrix}
\begin{bmatrix}
w \\
q
\end{bmatrix} + 
\begin{bmatrix}
\frac{\tilde{Z}_\eta}{m - \tilde{Z}_w} \\
\frac{\tilde{M}_\eta}{I_y}
\end{bmatrix} \eta
\]

- If was assume: \( \tilde{Z}_q << mU_e \) and \( \tilde{Z}_w << m \)

\[
\begin{bmatrix}
\dot{w} \\
\dot{q}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\tilde{Z}_w}{m} & U_e \\
\frac{m}{\tilde{M}_w} & \frac{\tilde{M}_q}{I_y}
\end{bmatrix}
\begin{bmatrix}
w \\
q
\end{bmatrix} + 
\begin{bmatrix}
\frac{\tilde{Z}_\eta}{m} \\
\frac{\tilde{M}_\eta}{I_y}
\end{bmatrix} \eta
\]
Final approximation

- So that, finally, the natural frequency and damping ratio of the short period oscillation mode is given by:

\[
2\xi_s \omega_s = - \left( \frac{\tilde{M}_q}{I_y} + \frac{\tilde{Z}_w}{m} + \frac{\tilde{M}_w}{I_y} U_e \right)
\]

\[
\omega_s = \sqrt{\frac{\tilde{M}_q}{I_y} \frac{\tilde{Z}_w}{m} - \frac{\tilde{M}_w}{I_y} U_e}
\]

- Cruder approximation:

For conventional aircraft!

\[
2\xi_s \omega_s = - \frac{\tilde{M}_q}{I_y}
\]

\[
\omega_s = \sqrt{- \frac{\tilde{M}_w}{I_y} U_e}
\]
Short Period Dependencies

- The period generally decreases with airspeed. The damping can either decrease or increase.
- Compressibility effects.
- Period and damping for a Boeing 747 at several altitudes and Mach numbers.
Effect of static margin

Small static margins increase the period and increase the damping of Short Period Oscillations.
Spiral mode

• This mode is quite visible in the impulse response of the lateral equations
• It is the non-oscillatory mode with large time constant
• It is mainly a yaw movement with a little roll
• This mode can be stable or unstable. It is unstable quite often but that is not a problem because of its large time constant
• The typical half-life of a spiral mode is of the order of a minute.
• The spiral movement is usually stopped by a corrective control input
Spiral Mode Video
Roll subsidence

- An impulsive aileron input will start the aircraft rolling.
- In general, the aircraft will stop rolling with time (i.e. the roll rate becomes zero after sufficient time)
- The aircraft will find itself at a roll angle which depends on how fast the roll rate tends to zero.
- This phenomenon is called roll subsidence.
Roll Subsidence Video
Dutch Roll

• The name Dutch Roll is due to the fact that the phenomenon resembles an ice skating figure called Dutch Roll.

• The centre of gravity remains on a straight trajectory while the roll and yaw angles oscillate.

• The roll velocity also oscillates but the yaw velocity is very low.

• The Dutch roll damping increases with airspeed while its period first increases and then decreases with airspeed.

• The typical period of a Dutch roll is in the order of 5 to 10 seconds.
Dutch Roll graphic

Imagine all drawings on a single straight line
Dutch Roll Videos
Dutch Roll approximation

• There is little consensus on how to simplify the Dutch Roll mode.
• In fact there is little consensus on what the Dutch Roll mode involves for a generic airplane.
• Cook states that ‘it is probably true for most airplanes that the roll to yaw ratio is less than one’.
  – ‘… in some cases (it) may be much less than one’.
  – Using this assumption, a simplification of the equations of motion can be carried out.
• McCormick states that the roll and yaw motions have approximately the same magnitude. He backs it up with results from the Piper Cherokee aircraft.
  – He states that the Dutch Roll motion is characterized by roll, yaw and sideslip.
  – In this case, no real simplification can be carried out.
Cook’s Dutch Roll approximation

• Use the longitudinal equations of motion with the assumptions:

\[ \dot{p} = p = \dot{\phi} = \phi = 0 \]

• So that,

\[
\begin{bmatrix}
\dot{v} \\
\dot{r}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Y}_v & V_0 \\
\frac{m}{\tilde{N}_v} & \frac{\tilde{N}_r}{I_z} \\
\frac{\tilde{N}_v}{I_z} & \frac{\tilde{N}_r}{I_z}
\end{bmatrix}
\begin{bmatrix}
v \\
r
\end{bmatrix}
\]

• and

\[
2\xi_d \omega_d = -\left(\frac{\tilde{N}_r}{I_z} + \frac{\tilde{Y}_v}{m}\right)
\]

\[
\omega_s = \sqrt{\frac{\tilde{N}_r \tilde{Y}_v}{I_z m} - \frac{\tilde{N}_v}{I_z} V_0}
\]
### Lateral Modes of Boeing 747

<table>
<thead>
<tr>
<th>Altitude (ft)</th>
<th>Mach</th>
<th>Spiral Half-life</th>
<th>Dutch Roll Period</th>
<th>$N_{\text{half}}$</th>
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<tr>
<td>0</td>
<td>0.45</td>
<td>35.7</td>
<td>5.98</td>
<td>0.87</td>
</tr>
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<td>0</td>
<td>0.65</td>
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<td>4.54</td>
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<td>0.5</td>
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<td>7.3</td>
<td>1.58</td>
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<td>5.89</td>
<td>1.33</td>
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<tr>
<td>20,000</td>
<td>0.8</td>
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<td>0.9</td>
<td>-89.2</td>
<td>6.19</td>
<td>1.18</td>
</tr>
</tbody>
</table>
Stability Analysis

- The stability of the equations of motion is analysed as usual
- The stability depends on the eigenvalues of the matrix $A$
- If all of the eigenvalues have negative real parts the system is stable
- If at least one eigenvalue has a positive real part the system is unstable
- If at least one of the eigenvalues has a zero real part the system is neutrally stable
Lateral Equations stability

- The lateral equations are usually almost neutrally stable.
- Example: F4 lateral equations have these eigenvalues: -0.1602 + 1.8141i, -0.1602 - 1.8141i, -0.6506, -0.0172, -0.000.
- The -0.000 eigenvalue is due to the fact that aircraft have little or no restoring force in the roll direction.
- Fighter aircraft, like the F4, are designed to have no restoring force in roll and some aircraft, like the F104, were designed to be unstable in roll.