Flight Dynamics and Control

Lecture 5:
Longitudinal stability
Derivatives

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Previously on AERO0003-1

- We developed linearized equations of motion

**Longitudinal direction**

\[
\begin{bmatrix}
  m & 0 & 0 & 0 & 0 \\
  0 & I_x & -I_{xz} & 0 & 0 \\
  0 & -I_{xz} & I_x & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  \dot{v} \\
  \dot{p} \\
  \dot{r} \\
  \dot{\phi} \\
  \dot{\psi} \\
\end{bmatrix}
+ \begin{bmatrix}
  -\ddot{v} \\
  -\ddot{p} \\
  -\ddot{r} \\
  0 \\
  0 \\
\end{bmatrix}
= \begin{bmatrix}
  -\dot{Y}_v - (\dot{Y}_p + mW_e) - (\dot{Y}_r - mU_e) \\
  -\dot{L}_v - \dot{L}_p - \dot{L}_r \\
  -\dot{N}_v - \dot{N}_p - \dot{N}_r \\
  0 \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  v \\
  p \\
  r \\
  \varphi \\
  \psi \\
\end{bmatrix}
\]

**Lateral direction**

\[
\begin{bmatrix}
  m & -\ddot{X}_w & 0 & 0 \\
  0 & (m - \ddot{Z}_w) & 0 & 0 \\
  0 & -\ddot{M}_w & I_y & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  \ddot{u} \\
  \ddot{w} \\
  \ddot{\theta} \\
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  0 \\
  0 \\
\end{bmatrix}
= \begin{bmatrix}
  -\ddot{X}_u - \ddot{X}_w - (\ddot{X}_q - mW_e) \\
  -\ddot{Z}_u - \ddot{Z}_w - (\ddot{Z}_q + mU_e) \\
  -\ddot{M}_u - \ddot{M}_w - \ddot{M}_q \\
  0 \\
\end{bmatrix}
\begin{bmatrix}
  u \\
  w \\
  q \\
  \theta \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \ddot{Y}_v \\
  \ddot{L}_v \\
  \ddot{N}_v \\
  0 \\
\end{bmatrix}
= \begin{bmatrix}
  \theta \\
  \phi \\
  \psi \\
\end{bmatrix}
\]
Longitudinal stability derivatives

• It has already been stated that the best way to obtain the values of the stability derivatives is to measure them.
• However, it is still useful to discuss simplified methods of estimating these coefficients.
• Such estimates can be used, for example, in the preliminary design of aircraft.
• This lecture will treat longitudinal stability derivatives.
Simple example

• We keep the quasi-steady aerodynamic assumption.
• Assume that the lift of an aircraft lies entirely in the $z$ direction:

$$ Z = \frac{1}{2} \rho U^2 S C_L $$

• where $C_L$ is the lift coefficient, assumed to be constant in this simple example.
• Now consider that there is a perturbation in the airspeed such that $U = U_e + u$, $U_e$ being the equilibrium airspeed
Simple example (2)

• The total aerodynamic force in the $z$ direction becomes:

$$Z = \frac{1}{2} \rho (U_e + u)^2 SC_L = \frac{1}{2} \rho (U_e^2 + 2U_e u + u^2) SC_L$$

• We linearize such that $u^2 = 0$:

$$Z = \frac{1}{2} \rho (U_e^2 + 2U_e u) SC_L$$

• Now we can calculate the derivative of $Z$ with respect to $u$:

$$\tilde{Z}_u = \frac{\partial Z}{\partial u} = \rho U_e SC_L$$
Lift and Drag

• This simple example considered only one aerodynamic force and one perturbation. Now we need to consider all aerodynamic loads and all perturbations.

• Lift and drag are defined in directions perpendicular and parallel to the direction of motion:

![Diagram showing Lift and Drag forces and moments]
Longitudinal geometry and loads

\[
\bar{V}_T = \frac{S_T l_T}{S \bar{c}}
\]

\( h \) denotes the cg position

\( h_0 \) denotes the aerodynamic centre

From Aircraft Design Lectures
Z\textsubscript{a}, X\textsubscript{a} and M\textsubscript{a}

- Clearly, the aerodynamic forces Z\textsubscript{a} and X\textsubscript{a} and the aerodynamic moment M\textsubscript{a} can be written as:

  \[ Z\textsubscript{a} = -\text{Lift} \times \cos \alpha - \text{Drag} \times \sin \alpha \]
  \[ X\textsubscript{a} = \text{Lift} \times \sin \alpha - \text{Drag} \times \cos \alpha \]
  \[ M\textsubscript{a} = M\textsubscript{0} + \text{Lift} (h - h\textsubscript{0}) \bar{c} - \text{Lift}_T l_T \]

- The last equation was obtained during the Aircraft Design lectures.

- We have assumed that the wing lift is approximately equal to the total lift.
In coefficient form

• The lift and drag can be written, as usual, in terms of the lift and drag coefficients, so that

\[ Z_a = -\frac{1}{2} \rho Q_\infty^2 S \left( C_L \cos \alpha + C_D \sin \alpha \right) \]

\[ X_a = \frac{1}{2} \rho Q_\infty^2 S \left( C_L \sin \alpha - C_D \cos \alpha \right) \]

\[ M_a = \frac{1}{2} \rho Q_\infty^2 Sc \left( C_{m_0} + C_L \left( h - h_0 \right) - \bar{V}_T C_L T \right) \]

• Using the notation of Aircraft Design,

\[ C_L = C_{L_0} + a \alpha \quad a = \text{lift curve slope} \]

\[ C_D = C_{D_0} + \frac{C_L^2}{e \pi AR} \quad e = \text{Oswald factor} \]
Angle of attack

- Consider an aircraft in equilibrium at angle of attack $\alpha_e$, perturbed by small airspeeds $u$, $w$ and small pitch rate $q$.

\[
\alpha = \tan^{-1} \left( \frac{W_e + w}{U_e + u} \right) \approx \tan^{-1} \left( \frac{W_e}{U_e} \right) + \tan^{-1} \left( \frac{w}{U_e} \right)
\]

or

\[
\alpha = \alpha_e + \frac{w}{U_e}
\]
Effective camber

• A pitching wing can be seen to feature an additional camber distribution due to the pitching motion.

\[h_0 = \frac{\bar{c}}{4}\]

\[h_1 = \frac{3\bar{c}}{4}\]

Local flow velocity due to pitch rate

Additional contribution to angle of attack

\[\alpha_{\text{eff}} = \left( h_1 - h \right) \frac{q}{U_e}\]
Wing-tail flow geometry

- For a static aircraft, the downwash effect of the wing deflects the free stream flow seen by the tailplane by an angle $\alpha$.
- Total angle of attack of tail: $\alpha_T = \alpha - \varepsilon + \eta_T$
- The total lift on the tailplane is given by:

$$C_{L_T} = a_0 + a_1 \alpha_T + a_2 \eta + a_3 \beta \eta$$
Unsteady wing downwash

• Now, consider a dynamic aircraft. A change in downwash at the wing because of a change in pitch angle at time $t$ will at the tail $\Delta t$ seconds later.

• The time difference $\Delta t$ is the time it takes for air leaving the wing to reach the tail and can be approximated as $\Delta t = l_t / U_e$.

• Then:

$$\varepsilon(t) = \frac{\partial \varepsilon}{\partial \alpha} \alpha - \frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \Delta t$$

• So that:

$$C_{LT} = a_1 \eta_T + a_1 \left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) \alpha + a_1 \frac{\partial \varepsilon}{\partial \alpha} \frac{l_t}{U_e} q + a_2 \eta$$
Lift, drag and moment

- In terms of perturbations around a static equilibrium condition, the lift, drag and moment can be written as:

\[ C_L = C_{Le} + \frac{a}{U_e} (w + (h_1 - h)q) \]

\[ C_D = C_{De} + \frac{2aC_{Le}}{U_e e\pi AR} (w + (h_1 - h)q) \]

\[ C_m = C_{me} + \left( h - h_0 - \bar{V}_T \frac{a}{a_1} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} (w + (h_1 - h)q) - \bar{V}_T a_1 \frac{\partial \epsilon}{\partial \alpha} \frac{l}{U_e} q - \bar{V}_T a_2 \eta \]

Second order perturbation terms in the drag expression have been ignored
Added mass terms

• The acceleration stability derivatives, e.g. $M_w$, $X_w$, $Z_w$ result from the added mass effect of the aircraft’s motion.

• This motion carries along a certain mass of air around the aircraft.

• As a consequence, the aircraft has more inertia than it would have had if it was moving in a vacuum.

• This is known as the added mass or apparent mass effect.
Added mass terms (2)

• Added mass can be seen as the inertia of cylinder of diameter $\overline{c}$ and length $b$ that is moving with the wing.

• Some added mass terms:

$$Lift_{added} = \rho \pi \frac{\overline{c}}{4} S (\dot{w} + U_e q)$$

$$Moment_{added} = \rho \pi \frac{\overline{c}}{4} S \left( h - \frac{\overline{c}}{2} \right) \dot{w} - \rho \pi \frac{\overline{c}}{4} S \left( \frac{3\overline{c}}{4} - h \right) U_e q - \frac{\pi}{16} \rho U_e \overline{c}^2 S q$$
Perturbation

• Consider that, as the flight is perturbed, the total airspeed is:

\[ Q_x^2 = (U_e + u)^2 + (W_e + w)^2 = V_0^2 + 2U_e u + 2W_e w \]

• The cosine and sine terms of the angle of attack become:

\[
\cos \alpha = \cos \left( \alpha_e + \frac{w}{U_e} \right) = \cos \alpha_e - \sin \alpha_e \frac{w}{U_e}
\]

\[
\sin \alpha = \sin \left( \alpha_e + \frac{w}{U_e} \right) = \sin \alpha_e + \cos \alpha_e \frac{w}{U_e}
\]
Force $Z_a$

- The total force $Z_a$ becomes:

$$Z_a = -\frac{1}{2} \rho Q_x^2 S (C_{Le} \cos \alpha + C_{De} \sin \alpha) - \frac{1}{2} \rho Q_x^2 S \left( \frac{a}{U_e} (w + (h_1 - h)q) \cos \alpha + \frac{2aC_{Le}}{U_e e \pi AR} \left( w + (h_1 - h)q \right) \sin \alpha \right)$$

$$- \rho \pi \bar{c} S (\dot{w} + U_e q) \cos \alpha$$

- After linearization:

$$Z_a = Z_{ae} - \rho (U_e u + W_e w) S (C_{Le} \cos \alpha_e + C_{De} \sin \alpha_e) - \frac{1}{2U_e} \rho V_0^2 S \left( -C_{Le} \sin \alpha_e + C_{De} \cos \alpha_e \right) w$$

$$- \frac{1}{2} \rho V_0^2 S \left( \frac{a}{U_e} (w + (h_1 - h)q) \cos \alpha_e + \frac{2aC_{Le}}{U_e e \pi AR} \left( w + (h_1 - h)q \right) \sin \alpha_e \right) - \rho \pi \bar{c} S (\dot{w} + U_e q) \cos \alpha_e$$
Z stability derivatives

• Differentiating $Z_a$ with respect to $w$, $u$, $q$ etc yields the Z stability derivatives as:

\[
\begin{align*}
\tilde{Z}_u &= -\rho U_e S \left( C_L e \cos \alpha_e + C_D e \sin \alpha_e \right) \\
\tilde{Z}_w &= -\rho W_e S \left( C_L e \cos \alpha_e + C_D e \sin \alpha_e \right) - \frac{1}{2U_e} \rho V_0^2 S \left( -C_L e \sin \alpha_e + C_D e \cos \alpha_e \right) \\
&\quad - \frac{1}{2} \rho V_0^2 S \left( \frac{a}{U_e} \cos \alpha_e + \frac{2aC_L e}{U_e e \pi AR} \sin \alpha_e \right) \\
\tilde{Z}_q &= -\frac{1}{2} \rho V_0^2 S \left( \frac{a}{U_e} (h_1 - h) \cos \alpha_e + \frac{2aC_L e}{U_e e \pi AR} (h_1 - h) \sin \alpha_e \right) - \rho \pi \frac{c}{4} S U_e \cos \alpha_e \\
\tilde{Z}_w &= -\rho \pi \frac{c}{4} S \cos \alpha_e
\end{align*}
\]
Force $X_a$

- The total force $X_a$ becomes:

$$X_a = \frac{1}{2} \rho Q^2 \left( C_{L_e} \sin \alpha - C_{D_e} \cos \alpha \right) + \frac{1}{2} \rho Q^2 \left( \frac{a}{U_e} (w + (h_1 - h)q) \sin \alpha - \frac{2a C_{L_e}}{U_e e \pi AR} (w + (h_1 - h)q) \cos \alpha \right)$$

$$+ \rho \pi \frac{c}{4} S (\dot{w} + U_eq) \sin \alpha$$

- After linearization:

$$X_a = X_{ae} + \rho (U_eq + W_eq) S \left( C_{L_e} \sin \alpha_e - C_{D_e} \cos \alpha_e \right) + \frac{1}{2} \rho V^2 S \left( C_{L_e} \cos \alpha_e + C_{D_e} \sin \alpha_e \right) w$$

$$+ \frac{1}{2} \rho V_0^2 \left( \frac{a}{U_e} (w + (h_1 - h)q) \sin \alpha_e - \frac{2a C_{L_e}}{U_e e \pi AR} (w + (h_1 - h)q) \cos \alpha_e \right) + \rho \pi \frac{c}{4} S (\dot{w} + U_eq) \sin \alpha_e$$
$X$ stability derivatives

- Differentiating $X_a$ with respect to $w$, $u$, $q$ etc yields the $X$ stability derivatives as:

\[
\begin{align*}
\tilde{X}_u &= \rho U_e S \left( C_{L_e} \sin \alpha_e - C_{D_e} \cos \alpha_e \right) \\
\tilde{X}_w &= \rho W_e S \left( C_{L_e} \sin \alpha_e - C_{D_e} \cos \alpha_e \right) + \frac{1}{2U_e} \rho V^2 S \left( C_{L_e} \cos \alpha_e + C_{D_e} \sin \alpha_e \right) \\
&\quad + \frac{1}{2} \rho V^2 S \left( \frac{a}{U_e} \sin \alpha_e - \frac{2aC_{L_e}}{U_e e \pi AR} \cos \alpha_e \right) \\
\tilde{X}_q &= \frac{1}{2} \rho V^2 S \left( \frac{a}{U_e} (h_1 - h) \sin \alpha_e - \frac{2aC_{L_e}}{U_e e \pi AR} (h_1 - h) \cos \alpha_e \right) + \rho \pi \frac{\bar{c}}{4} S U_e \sin \alpha_e \\
\tilde{X}_w &= \rho \pi \frac{\bar{c}}{4} S \sin \alpha_e
\end{align*}
\]
Moment $M_a$

- The total moment $M_a$ becomes:

$$M_a = \frac{1}{2} \rho Q^2 \bar{S} \bar{C} m_e + \frac{1}{2} \rho Q^2 \bar{S} \bar{C} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} \left( w + (h_1 - h) q \right)$$

$$- \frac{1}{2} \rho Q^2 \bar{S} \bar{C} \bar{V}_T a_1 \frac{\partial \varepsilon}{\partial \alpha} \frac{l_t}{U_e} q - \frac{1}{2} \rho V^2 \bar{S} \bar{C} \bar{V}_T a_2 \eta$$

$$+ \rho \pi \frac{\bar{c}}{4} S \left( h - \frac{\bar{c}}{2} \right) \dot{w} - \rho \pi \frac{\bar{c}}{4} S \left( \frac{3\bar{c}}{4} - h \right) U_e q - \frac{\pi}{16} \rho U_e \bar{c}^2 S q$$

- After linearization:

$$M_a = M_a + \rho \left( U_e u + W_e w \right) \bar{S} \bar{C} m_e + \frac{1}{2} \rho V^2 \bar{S} \bar{C} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} \left( w + (h_1 - h) q \right)$$

$$- \frac{1}{2} \rho V^2 \bar{S} \bar{C} \bar{V}_T a_1 \frac{\partial \varepsilon}{\partial \alpha} \frac{l_t}{U_e} q - \frac{1}{2} \rho V^2 \bar{S} \bar{C} \bar{V}_T a_2 \eta$$

$$+ \rho \pi \frac{\bar{c}}{4} S \left( h - \frac{\bar{c}}{2} \right) \dot{w} - \rho \pi \frac{\bar{c}}{4} S \left( \frac{3\bar{c}}{4} - h \right) U_e q - \frac{\pi}{16} \rho U_e \bar{c}^2 S q$$
$M$ stability derivatives

- Differentiating $M_a$ with respect to $w, u, q$ etc yields the $M$ stability derivatives as:

\[ M_u = \rho U_e \overline{S} \overline{c} C_{m_e} \]
\[ M_w = \rho W_e \overline{S} \overline{c} C_{m_e} + \frac{1}{2} \rho V_0^2 \overline{S} \overline{c} \left( h - h_0 - \overline{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} \]
\[ M_q = \frac{1}{2} \rho V_0^2 \overline{S} \overline{c} \left( h - h_0 - \overline{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} (h_1 - h) \]
\[ -\frac{1}{2} \rho V_0^2 \overline{S} \overline{c} \overline{V}_T a_1 \frac{\partial \varepsilon}{\partial \alpha} \frac{l_i}{U_e} - \rho \pi \overline{c} \left( \frac{3 \overline{c}}{4} - h \right) \frac{4}{16} \rho U_e \overline{c}^2 S \]
\[ M_{\eta} = \rho \pi \overline{c} S \left( h - \frac{\overline{c}}{2} \right) \]
\[ M_{\eta} = -\frac{1}{2} \rho V_0^2 \overline{S} \overline{c} \overline{V}_T a_2 \]
Discussion of longitudinal derivatives

• Notice that this methodology for the stability derivatives is by no means exact.
• Additional contributions need to be considered e.g.:
  – Aerodynamic pitching moment acting on the fuselage, including added mass terms.
  – At high subsonic airspeeds, \( u \) leads to an increase in Mach number and, hence, an additional increase in drag.
• By assuming that Total Lift=Wing Lift, we have neglected the effect of the elevator on \( X_a \) and \( Z_a \). The stability derivatives \( X_\eta \) and \( Z_\eta \) were not evaluated.
Equations of motion

- Remember that the stability derivatives must be inserted into the equations of motion:

\[
\begin{bmatrix}
m & -\ddot{X}_w & 0 & 0 \\
0 & (m - \ddot{Z}_w) & 0 & 0 \\
0 & -\dddot{M}_w & I_y & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\ddot{w} \\
\dddot{q} \\
\dot{\theta} \\
\end{bmatrix}
+ \begin{bmatrix}
-\ddot{X}_u & -\ddot{X}_w & -(\dddot{X}_q - mW_e) & mg \cos \theta_e \\
-\ddot{Z}_u & -\ddot{Z}_w & -(\dddot{Z}_q + mU_e) & mg \sin \theta_e \\
-\dddot{M}_u & -\dddot{M}_w & -\dddot{M}_q & 0 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
q \\
\theta \\
\end{bmatrix}
= \begin{bmatrix}
\dddot{X}_\eta & \dddot{X}_\tau \\
\dddot{Z}_\eta & \dddot{Z}_\tau \\
\dddot{M}_\eta & \dddot{M}_\tau \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\eta \\
\tau \\
\end{bmatrix}
\]
Lateral stability derivatives

- Lateral stability derivatives can be obtained following a similar analysis of the lateral degrees of freedom of the aircraft.
- Added mass terms are important for wing roll and fin yaw.
  - Fuselage yaw and fuselage sideslip added mass terms are also important but difficult to model.
- Effective camber is important for the fin lift and moment.
- Unsteadiness is important in the calculation of the fuselage sidewash effect on the fin but is very difficult to model.
Unsteady aerodynamics

• In principle, the quasi-steady aerodynamic approach followed here is inaccurate.
• It mostly ignores the effect that the aircraft’s wake has on the aircraft itself.
• In the present analysis only one particularly important unsteady effect was modelled.
• Other effects may be of importance. Nevertheless, it is assumed that flight dynamic motion happens at frequencies low enough to generally neglect unsteadiness.
• In the domain of aeroelasticity, the frequencies of oscillation are much higher and unsteadiness cannot be neglected.
Static stability

• Now we can revisit static stability, as discussed in the Aeronautical Design lectures

• Recall that the linearized moment equation is:

\[
M_a = M_{ae} + \rho (U_e u + W_e w) S \bar{c} C_{m_e} + \frac{1}{2} \rho V_0^2 \bar{S} \bar{c} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \frac{a}{U_e} (w + (h_1 - h) q)
\]

\[- \frac{1}{2} \rho V_0^2 S \bar{c} \bar{V}_T a_1 \frac{\partial \varepsilon}{\partial \alpha} \frac{l_t}{U_e} q - \frac{1}{2} \rho V_0^2 S \bar{c} \bar{V}_T a_2 \eta
\]

\[+ \rho \pi \frac{\bar{c}}{4} S \left( h - \frac{\bar{c}}{2} \right) \dot{w} - \rho \pi \frac{\bar{c}}{4} S \left( \frac{3 \bar{c}}{4} - h \right) U_e q - \frac{\pi}{16} \rho U_e \bar{c}^2 S q \]
Static stability (2)

- Static conditions means that all kinematic conditions are constant.
- This means that all perturbations are zero: $u = w = q = \eta = 0$
- We will consider a small static change in angle of attack $\alpha = \alpha_e + \Delta \alpha$.
- Only the third term in the moment equation depends directly on angle of attack. Remember that we approximated:

$$\alpha = \alpha_e + \frac{w}{U_e}$$

- So that

$$\Delta \alpha = \frac{w}{U_e}$$
Static stability (3)

- Substituting this angle of attack perturbation in the third terms of the moment equation and neglecting all other terms:

\[ M_a = M_{ae} + \frac{1}{2} \rho V_0^2 S \bar{c} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) a \Delta \alpha \]

- After non-dimensionalisation:

\[ C_{M_a} = C_{M_{ae}} + \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) a \Delta \alpha \]
Static stability (4)

• Note that the product $a\Delta \alpha$ an incremental lift coefficient $\Delta C_L = a\Delta \alpha$.

• The moment coefficient equation becomes:

$$C_{M_a} = C_{M_{ae}} + \left( h - h_0 - \frac{V_T}{a} a_1 \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right) \Delta C_L$$

• The stability of an aircraft is defined as the rate of change of the moment coefficient with the lift coefficient

$$\frac{dC_{M_a}}{d\Delta C_L} = \left( h - h_0 - \frac{V_T}{a} a_1 \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right)$$
Controls fixed stability

• The controls fixed stability margin is defined as

\[ K_n = -\frac{dC_{Ma}}{d\Delta C_L} \]

• Or

\[ K_n = h_0 + \bar{V}_T \frac{a_1}{a} \left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) - h \]

• We can also defined the controls fixed neutral point such that

\[ h_n = h_0 + \bar{V}_T \frac{a_1}{a} \left(1 - \frac{d \varepsilon}{d \alpha}\right) \quad \text{and} \quad K_n = h_n - h \]
Controls fixed stability (2)

• Note that the controls fixed stability margin is related to the $M_w$ aerodynamic stability derivative:

$$\tilde{M}_w = \rho W_e S \bar{c} C_{m_e} + \frac{1}{2} \rho V_0^2 S \bar{c} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right) \frac{a}{U_e}$$

• Substituting from the definition of $K_n$:

$$\tilde{M}_w = \rho W_e S \bar{c} C_{m_e} - \frac{1}{2} \rho V_0^2 S \bar{c} K_n \frac{a}{U_e}$$
Stability margin

- A stable aircraft has positive stability margin. The more positive, the more stable.
- If the cg position \((h)\) is ahead of the neutral point \((h_n)\) the aircraft will by definition be stable.
- Certification authorities specify that
  \[ K_n \geq 0.05c \]
  at all times.
- Of course, the stability margin can change:
  - If fuel is used up
  - If payload is released:
    - Bombs
    - Missiles
    - External fuel tanks
    - Paratroopers
    - Anything else you can dump from a plane
Degree of stability

Too much stability can be a bad thing!
Controls free stability

- If the controls are free, then $\eta \neq 0$. The moment equation becomes

$$M_a = M_{ae} + \frac{1}{2} \rho V_0^2 S \overline{c} \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \Delta C_L - \frac{1}{2} \rho V_0^2 S \overline{c} \bar{V}_T a_2 \eta$$

- Or

$$C_{Ma} = C_{Ma_e} + \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) \Delta C_L - \bar{V}_T a_2 \eta$$

- The rate of change of the moment coefficient with lift becomes

$$\frac{\partial C_{Ma}}{\partial \Delta C_L} = \left( h - h_0 - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right) - \bar{V}_T a_2 \frac{\partial \eta}{\partial \Delta C_L}$$
Elevator Hinge Moment (1)

• The rate of change of $h$ with incremental lift is an unknown.
• Consider the aerodynamic moment of the elevator around its hinge:
Elevator Hinge Moment (2)

• Since the elevator is free to rotate, the elevator hinge moment must be equal to zero.

• Assuming small displacement, the elevator hinge moment is a linear function of total angle of attack, elevator angle and trim tab angle, exactly like the lift. Therefore:

\[ C_H = b_1 \alpha_T + b_2 \eta + b_3 \beta \eta = 0 \]

• Where \( b_1 \), \( b_2 \) and \( b_3 \) are known constants

• Substituting for \( \alpha_T \) and solving for the elevator angle gives

\[ \eta = -\frac{\Delta C_L}{a} \left( \frac{b_1}{b_2} \right) \left( 1 - \frac{d \epsilon}{d \alpha} \right) - \frac{b_3}{b_2} \beta \eta - \frac{b_1}{b_2} \eta_T \]
Controls Free Stability Margin

• Assuming that the trim tab angle is constant

\[
\frac{\partial \eta}{\partial \Delta C_L} = -\frac{1}{a} \frac{b_1}{b_2} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right)
\]

• Substituting into the controls free stability equation:

\[
\frac{\partial C_{M_a}}{\partial \Delta C_L} = (h - h_0) - \bar{V}_T \frac{a_1}{a} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \left( 1 - \frac{a_2 b_1}{a_1 b_2} \right)
\]

• Define the Controls Free Stability Margin, \( K'_n \), such that

\[
K'_n = - \frac{\partial C_{M_a}}{\partial \Delta C_L} = h'_n - h
\]

• With \( h'_n \) being the controls free neutral point
Controls Free Neutral Point

• The controls free neutral point is then

\[ h'_n = h_0 + \frac{V_T}{a} a_1 \left( 1 - \frac{d\varepsilon}{d\alpha} \right) \left( 1 - \frac{a_2 b_1}{a_1 b_2} \right) \]

• Using the expression for the controls fixed neutral point \( h_n = h_0 + \frac{V_T}{a} a_1 \left( 1 - \frac{d\varepsilon}{d\alpha} \right) \) gives

\[ h'_n = h_n - \frac{V_T}{a b_2} a_2 b_1 \left( 1 - \frac{d\varepsilon}{d\alpha} \right) \]
Discussion

• As with the controls fixed stability margin, the controls free stability margin is positive when the aircraft is stable.
• Similarly, the centre of gravity position must be ahead of the controls free neutral point if the aircraft is to be stable.
• Usually, the constants of the elevator and tab are such that $h'_n > h_n$.
• An aircraft that is stable controls fixed will usually be also stable controls free.
Summary of Longitudinal Stability